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Quantizable forms ★

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Abstract

The contact manifolds that appear in Geometric Quantization can be obtained from certain elements of the dual of the Lie algebra. Many of the known methods to determine these invariant forms are revisited and sharpened.

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1. Introduction

The starting point in Geometric Quantization (Kirillov–Kostant–Souriau theory, cf. [7,9]) is a regular contact manifold whose Boothby–Wang fibration [1] has a covering of a coadjoint orbit of a given Lie group as base space. In this paper we are concerned with the case in which the group acts transitively on the contact manifold by diffeomorphisms that preserves the contact form (Homogeneous Contact Manifolds). More specifically, the paper is devoted to give explicit methods of construction of such manifolds. In particular many of the previously known related results [3,4,8,10] are covered and some of them are sharpened.

We first consider the following slightly more general situation. Let M be a differentiable manifold and G a Lie group which acts transitively on M. Let us consider a Pfaff system on M, composed of invariant forms, whose characteristic system is trivial. Such a differential system is called homogeneous nondegenerate Pfaff system.

In the present paper we classify the homogeneous nondegenerate Pfaff systems (HNDPS). This classification is constructive, in the sense that all HNDPS for a given Lie group can be

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explicitly determined from subsets of the dual of its Lie algebra. We will also see that they give rise to principal fibre bundles whose base spaces are multihamiltonian spaces. These are generalizations of Hamiltonian spaces, and are classified in this paper. When restricted to Hamiltonian spaces, this classification is equivalent to that of Kostant.

These general results are then applied to the case in which the Pfaff system consists of a single form, thus gaining an insight into the homogeneous contact manifolds. These can be obtained from certain elements of the dual of the Lie algebra of the given Lie group that are called quantizable forms.

The structure of the present paper is the following. Sections 2 and 3 are devoted to notation and some preliminary results.

The first statement of the classification is given in Section 4. Given a connected Lie group G, and a subset, P, of the dual of its Lie algebra, we associate to P a family of subgroups of G, N_P^j. The elements of P project on each G/N_P^j, thus giving rise to a HNDPS. It is proved that each HNDPS is equivalent, in a natural way, to one of these.

If one wants to use such a classification for an explicit construction of the HNDPS, the main difficulty is to determine the groups N_P^j. In Section 5 we prove that the N_P^j are the kernels of a family of homomorphisms, thus obtaining practical methods to determine the N_P^j. This idea leads to another form of the classification.

Multihamiltonian spaces are defined and classified in Section 6. Its main interest, from the point of view of the present paper, is that any "regular" HNDPS is the total space of a principal fibre bundle, with abelian structural group, whose base space is a multihamiltonian space. These principal fibre bundles and the connections that the given HNDPS defines on it are studied in detail in Section 7.

The explicit methods that can lead to the determination of the quantizable forms (i.e. the homogeneous contact manifolds) of a given Lie group are given in Section 8. This section also includes other related results.

In Section 9, the preceding results are applied to the universal covering group of Poincaré group, whose homogeneous contact manifolds describe the relativistic elementary particles in the sense of geometric quantization.

2. Notation

All differentiable manifolds appearing in this paper are assumed to be C^∞, finite dimensional, Hausdorff and second countable.

The set consisting of the differentiable vector fields on a differential manifold, M, is denoted by $\mathcal{D}(M)$. The set of differential k-forms on M is denoted by $\Omega^k(M)$.

Let $X \in \mathcal{D}(M)$, $\omega \in \Omega^k(M)$. We denote by $i(X)\omega$ the interior product of X by ω and by $L(X)\omega$ the Lie derivative of ω with respect to X.

Let G be a Lie group. The set consisting of the left invariant vector fields on G, provided with its canonical structure of Lie algebra, will be denoted by $\underline{G}$. We shall denote by $\underline{G}^*$ the dual of $\underline{G}$. $\underline{G}^*$ is canonically identified with the set consisting of the left invariant 1-forms on G.

The coadjoint representation is the homomorphism $Ad^* : g \in G \to Ad^*_g \in Aut(\underline{G}^*)$ given by $(Ad^*_g(\alpha))(X) = \alpha(Ad_{g^{-1}}(X))$ for all $\alpha \in \underline{G}^*$, $X \in \underline{G}$.

Let M be a differentiable manifold and G a Lie group acting on M on the left (resp. on the right). Given an element, X, of $\underline{G}$, we denote by X_M the vector field on M whose flow is given by (the diffeomorphisms associated by the action to) $\{Exp(-tX): t \in \mathbb{R}\}$ (resp. $\{Exp\, tX: t \in \mathbb{R}\}$). X_M will be called the infinitesimal generator of the action associated to X.

A principal fibre bundle having M as total space, B as base and G as structural group, will be denoted by $M(B, G)$.

3. Homogeneous nondegenerate systems

Let M be a differentiable manifold. We call *k-system* on M any set consisting of differential k-forms on M. A k-system on M, S, will be said to be *nondegenerate* when, for all $x \in M$, the set consisting of those $v \in T_xM$ such that $i(v)\omega = 0$ and $i(v)\,\mathrm{d}\omega = 0$ for all $\omega \in S$, consists of 0 alone.

Let M be a contact (resp. symplectic) manifold. If S is a 1-system (resp. 2-system) containing a contact (resp. symplectic) form, then S is nondegenerate.

Let S be a nondegenerate 1-system on M. We denote by $A(S)$ the set consisting of $X \in D(M)$ such that $i(X)\,\mathrm{d}\omega = 0$ and $\mathrm{d}i(X)\omega = 0$, for all $\omega \in S$. The function $i(X)\omega$ is constant on each connected component of M for all $X \in A(S)$, $\omega \in S$.

Lemma 3.1. *$A(S)$ is an abelian Lie subalgebra of $\mathcal{D}(M)$.*

Proof. $A(S)$ is obviously a vector subspace of $\mathcal{D}(M)$ and, for all $X, Y \in A(S)$ and $\omega \in S$, we have

$$\begin{aligned} i([X,Y])\omega &= (L(X)i(Y) - i(Y)L(X))\omega \\ &= i(X)\,\mathrm{d}i(Y)\omega - i(Y)(\mathrm{d}i(X) + i(X)\mathrm{d})\omega = 0 \end{aligned}$$

and, in a similar way, $i([X,Y])\,\mathrm{d}\omega = 0$. Thus, since S is nondegenerate, we have $[X, Y] = 0$. □

Example 3.2. Let ω be a contact form on a connected manifold, M, and $S = \{\omega\}$. It is a well known fact that there exists a unique $Z(\omega) \in \mathcal{D}(M)$ such that $i(Z(\omega))\omega = 1$, $i(Z(\omega))\,\mathrm{d}\omega = 0$. Thus $A(S)$ is the (1-dimensional) subspace of $\mathcal{D}(M)$ generated by $Z(\omega)$. In fact, if $A \in A(S)$ then $A = (i(A)\omega)Z(\omega)$.

A *G-homogeneous (nondegenerate) k-system* is a triple, (M, S, G), where M is a connected manifold, S a (nondegenerate) k-system on M and G is a connected Lie group acting transitively on M on the left by ω-preserving diffeomorphisms for all $\omega \in S$.

Lemma 3.3. *Let S be a 1-system on M consisting of a single 1-form, ω. Then S is nondegenerate if and only if ω is a contact form or* $\mathrm{d}\omega$ *is symplectic.*

Proof. The "if" part is obvious. In order to prove the "only if" part, take $x \in M$ and assume that $(\mathrm{d}\omega)_x^p \neq 0$, $(\mathrm{d}\omega)_x^{p+1} = 0$. Thus the subspace, V, of T_xM composed of v such that $i(v)(\mathrm{d}\omega)_x = 0$ has dimension $n - 2p$, where $n = \mathit{dim}\, M$. Let us denote by ω' the restriction of ω_x to V. Since S is nondegenerate it follows that the kernel of ω' must be $\{0\}$. Then we have $\mathit{dim}\, V \leq 1$, so that $2p \leq n \leq 2p + 1$, i.e. $n = 2p$ if n is even and $n = 2p + 1$ if n is odd. In any case p is independent of $x \in M$.

If $n = 2p$, then $(\mathrm{d}\omega)^p$ is a volume element, i.e. $\mathrm{d}\omega$ is symplectic.

If $n = 2p + 1$, then we have $\mathit{dim}\, V = 1$. Let u_1 be a nonzero element of V and $\{u_1, u_2, \ldots, u_n\}$ a basis of T_xM. Thus $\omega_x \wedge (\mathrm{d}\omega)_x^p(u_1, \ldots, u_n)$ coincides, up to a nonzero factor, with $[\omega_x(u_1)] \cdot [(\mathrm{d}\omega)_x^p(u_2, \ldots, u_n)]$. But $\omega_x(u_1) = \omega'(u_1) \neq 0$. On the other hand the relation $(\mathrm{d}\omega)_x^p \neq 0$ implies that there exists $i_1 \ldots i_{n-1} \in \{1 \ldots n\}$ such that $(\mathrm{d}\omega)_x^p(u_{i_1}, \ldots, u_{i_{n-1}}) \neq 0$. But this cannot be the case if any of the i were 1. Thus $(\mathrm{d}\omega)_x^p(u_2 \ldots u_n) \neq 0$. It follows that $\omega \wedge (\mathrm{d}\omega)^p$ is volume element, i.e. ω is a contact form. □

If S consists of a single differential form, ω, we denote (M, S, G) simply by (M, ω, G). A homogeneous nondegenerate 1-system (M, ω, G) where ω is a contact form is said to be a *Homogeneous Contact Manifold.* If $\mathrm{d}\omega$ is symplectic, then (M, ω, G) is said to be a *Homogeneous Exact Symplectic Manifold.*

Let (M, S, G) and (M', S', G) be G-homogeneous k-systems. An *isomorphism* from (M, S, G) onto (M', S', G) is a G-equivariant diffeomorphism, f, from M onto M', such that $f^*(S') = S$. This defines an equivalence relation in the set composed by the G-homogeneous k-systems. The equivalence class of (M, S, G) will be denoted by $|M, S, G|$. If an equivalence class has a nondegenerate representative, then any other representative is also nondegenerate. The set consisting of the equivalence classes of G-homogeneous nondegenerate k-systems will be denoted by $E_k(G)$.

In Section 4 we shall classify the set $E_1(G)$. The following result will be useful in this context.

Let $[M, S, G] \in E_1(G)$ and let $x \in M$. We shall denote by $\overline{x}$ the map from G onto M defined by sending each g to gx. We denote by G_x the isotropy subgroup of G at x.

Since each $\omega \in S$ is invariant by the action of G on M, it follows that $\overline{x}^*\omega$ is a left invariant 1-form on G. Thus $\overline{x}^*(S)$ can be considered as a subset of $\underline{G}^*$.

Lemma 3.4. *The Lie algebra of the isotropy subgroup of G at x, G_x, is*

$$\underline{G}_x = \{Y \in \underline{G} : i(Y)(\overline{x}^*\omega) = 0, i(Y)\,\mathrm{d}(\overline{x}^*\omega) = 0, \forall \omega \in S\}.$$

Proof. For all $Y, Z \in \underline{G}$, $\omega \in S$, we have

$$i(Y)(\overline{x}^*\omega) = (\overline{x}^*\omega)(Y) = (\overline{x}^*\omega)_e(Y_e) = \omega_x(-(Y_M)_x),$$

$$[i(Y)\,\mathrm{d}(\overline{x}^*\omega)](Z) = (\mathrm{d}(\overline{x}^*\omega))_e(Y_e, Z_e) = \mathrm{d}\omega_x(-(Y_M)_x, -(Z_M)_x).$$

Since the action is transitive, it follows that $T_x M = \{(Z_M)_x : Z \in \underline{G}\}$. Since S is nondegenerate, one thus sees that we have $(Y_M)_x = 0$ if and only if $i(Y)(\overline{x}^*\omega) = 0$ and $i(Y)\,\mathrm{d}(\overline{x}^*\omega) = 0$ for all $\omega \in S$. □

4. Classification of the homogeneous nondegenerate 1-systems

Let G be a connected Lie group. Let P be a subset of $\underline{G}^*$ containing some nonzero element. Thus we define

$$\underline{G}_P = \{X \in \underline{G} : L(X)\delta = 0, \forall \delta \in P\},$$
$$\underline{N}_P = \{X \in \underline{G}_P : i(X)\delta = 0, \forall \delta \in P\}.$$

Since P contains some nonzero element we have $\underline{N}_P \neq \underline{G}$.

Lemma 4.1. *$\underline{G}_P$ is a Lie subalgebra of $\underline{G}$, $\underline{N}_P$ is an ideal of $\underline{G}_P$ and $\underline{G}_P/\underline{N}_P$ is abelian.*

Proof. We shall prove that $[\underline{G}_P, \underline{G}_P] \subset \underline{N}_P$. If $X, Y \in \underline{G}_P$, $\delta \in P$, then we have

$$i([X, Y])\delta = L(X)i(Y)\delta - i(Y)L(X)\delta = 0$$

and

$$L([X, Y])\delta = L(X)L(Y)\delta - L(Y)L(X)\delta = 0.$$ □

Now we define

$$G_P = \{g \in G : Ad_g^*\delta = \delta, \forall \delta \in P\}.$$

Since we have

$$Ad_g^*\delta(Y) = \delta(Ad_{g^{-1}}Y) = \delta(R_{g*}L_{g^{-1}*}Y) = R_g^*\delta(Y)$$

for all $g \in G$, $\delta \in \underline{G}^*$, $Y \in \underline{G}$, it follows that

$$G_P = \{g \in G : R_g^*\delta = \delta, \forall \delta \in P\}.$$

G_P is a closed subgroup of G and its Lie algebra is composed of those $Y \in \underline{G}$ such that $R^*_{\mathrm{Exp}\,tY}\delta = \delta$ for all $\delta \in P, t \in \mathbb{R}$. Since the flow of the left invariant vector field Y is $\{R_{\mathrm{Exp}\,tY} : t \in \mathbb{R}\}$, it follows that the Lie algebra of G_P is $\underline{G}_P$. Obviously G_P contains the centre of G, hence $\underline{G}_P$ contains the centre of $\underline{G}$.

We shall denote by $\{G^i_P : i \in I(P)\}$ the set consisting of those subgroups of G_P containing the component of the identity in G_P. We assume that $0 \in I(P)$ and that G^0_P is the component of the identity in G_P. Thus G^0_P is the component of the identity in G^i_P for all $i \in I(P)$.

We shall denote by N^0_P the connected Lie subgroup of G whose Lie algebra is $\underline{N}_P$.

If N^0_P is closed, then we shall denote by $\{N^j_P : j \in J(P)\}$ the set consisting of the closed subgroups of G_P whose Lie algebra is $\underline{N}_P$. Here $J(P)$ is a set of indices containing 0. N^0_P is the component of the identity of each N^j_P and also an invariant subgroup of G^0_P.

Before proving the following lemma, note that if δ is a left invariant 1-form and X a left invariant vector field, then we have $L(X)\delta = i(X)\,\mathrm{d}\delta$. This is a consequence of the fact that $i(X)\delta$ is a constant function. Thus we have

$$\underline{G}_P = \{X \in \underline{G}\colon i(X)\mathrm{d}\delta = 0, \forall \delta \in P\}.$$

The following result is more or less obvious, but useful.

Lemma 4.2. *Let $\langle P\rangle$ be the subspace of $\underline{G}^*$ generated by P and let $g \in G$. Then we have $\underline{G}_{\langle P\rangle} = \underline{G}_P$, $\underline{N}_{\langle P\rangle} = \underline{N}_P$, $G_{\langle P\rangle} = G_P$, $N^0_{\langle P\rangle} = N^0_P$, $\underline{G}_{Ad^*_g(P)} = Ad_g(\underline{G}_P)$, $\underline{N}_{Ad^*_g(P)} = Ad_g(\underline{N}_P)$, $G_{Ad^*_g(P)} = a_g(G_P)$ and $\{N^{j'}_{Ad^*_g(P)}\colon j' \in J(Ad^*_g(P))\} = \{a_g(N^j_P)\colon j \in J(P)\}$.*

Let us denote by $K_0(G)$ the set consisting of the pairs (P, N^j_P), where P is such that N^0_P is closed and $j \in J(P)$.

In $K_0(G)$ we define an equivalence relation as follows. We say that (P, N^j_P) is equivalent to $(P', N^{j'}_{P'})$ if there exists $g \in G$ such that $P' = Ad^*_g(P)$ and $N^{j'}_{P'} = a_g(N^j_P)$. The equivalence class of (P, N^j_P) will be denoted by $|P, N^j_P|$ and the quotient set by $K(G)$.

We shall define a bijective map from $K(G)$ onto $E_1(G)$ but first we shall give some lemmas.

Lemma 4.3. *Let $(P, N^j_P) \in K_0(G)$. Then for all $\delta \in \langle P\rangle$, there exists a unique G-invariant 1-form, $\omega^j_P(\delta)$, on G/N^j_P such that $p^*_j(\omega^j_P(\delta)) = \delta$, where p_j is the canonical map from G onto G/N^j_P.*

Proof. Since $N^j_P \subset G_P$, δ is Ad_g-invariant for all $g \in N^j_P$. Also we have $i(X)\delta = 0$ for all $X \in \underline{N}_P$. □

For all $(P, N^j_P) \in K_0(G)$, we shall denote $P^j_1 = \{\omega^j_P(\delta)\colon \delta \in P\}$.

Lemma 4.4. *Let $(P, N^j_P) \in K_0(G)$. Then $(G/N^j_P, P^j_1, G)$ is a homogeneous nondegenerate 1-system for the canonical action of G on G/N^j_P.*

Proof. We only need to prove that P^j_1 is nondegenerate. Let v be a tangent vector to G/N^j_P at gN^j_P such that $i(v)(\omega^j_P(\delta)) = 0$ and $i(v)\,\mathrm{d}(\omega^j_P(\delta)) = 0$ for all $\delta \in p$. We must prove that $v = 0$. Since the action is transitive and the $\omega^j_P(\delta)$ are invariant, we only need to give a proof in the case where g is the identity element, e, of G. Let $X \in \underline{G}$ be such that the infinitesimal generator of the action associated to X, X^j, takes the value v at N^j_P. Then we have for all $Y \in G$

$$i(X)\delta = i(X_e)\delta_e = i(X_e)(p^*_j\omega^j_P(\delta))_e = i(-v)(\omega^j_P(\delta)) = 0,$$

$$(i(X)\,\mathrm{d}\delta)(Y) = \mathrm{d}\delta_e(X_e, Y_e) = \mathrm{d}(\omega^j_S(\delta))(v, (Y^j)_{N^j_P}) = 0.$$

Hence we have $X \in \underline{N}_P$ so that $v = (X^j)_{N_P^j} = 0$. □

Lemma 4.5. *Let $(P, N_P^j), (Q, N_Q^k) \in K_0(G)$. Then we have $|P, N_P^j| = |Q, N_Q^k|$ if and only if $|G/N_P^j, P_1^j, G| = |G/N_Q^k, Q_1^k, G|$.*

Proof. Assume that $|P, N_P^j| = |Q, N_Q^k|$. Let $g \in G$ be such that $Q = Ad_g^*(P)$, $N_Q^k = a_g(N_P^j)$. Then we define a map, b_g, from G/N_P^j into G/N_Q^k by sending hN_P^j to $hg^{-1}N_Q^k$. This is a well defined map. Since we have $p_k \circ R_{g^{-1}} = b_g \circ p_j$ and p_j admits differentiable local cross sections everywhere, it follows that b_g is differentiable. Actually b_g is a diffeomorphism whose inverse can be defined in a similar way.

On the other hand we have for all $\delta \in Q$,

$$p_j^* b_g^* \omega_Q^k(\delta) = R_{g^{-1}}^* p_k^* \omega_Q^k(\delta) = R_{g^{-1}}^* \delta = Ad_{g^{-1}}^* \delta$$

but $Ad_{g^{-1}}^* \delta \in P$, so that $p_j^* b_g^* \omega_Q^k(\delta) = p_j^* \omega_P^j(Ad_{g^{-1}}^* \delta)$. Since p_j is a submersion we obtain $b_g^* \omega_Q^k(\delta) = \omega_P^j(Ad_{g^{-1}}^* \delta)$. The relation $b_g^*(Q_1^k) = P_1^j$ follows.

Since b_g is G-equivariant for the canonical actions, we see that b_g is an isomorphism from $(G/N_P^j, P_1^j, G)$ onto $(G/N_Q^k, Q_1^k, G)$.

Conversely, let f be an isomorphism from $(G/N_P^j, P_1^j, G)$ onto $(G/N_Q^k, Q_1^k, G)$. Let $g \in G$ be such that $f(N_P^j) = gN_Q^k$.

The relation $Ad_g^*(Q) = P$ follows from $f \circ p_j = p_k \circ R_g$. On the other hand, because of the equivariance of f, the isotropy subgroup at $N_P^j \in G/N_P^j$, which is N_P^j, must coincide with the isotropy subgroup at gN_Q^k, which is $a_g(N_Q^k)$. Hence (Q, N_Q^k) is equivalent to (P, N_P^j). □

Now we define a map, μ_1, from $K(G)$ into $E_1(G)$ by sending $|P, N_P^j|$ to $|G/N_P^j, P_1^j, G|$. Because of Lemma 4.5 this is a well defined injective map. We now go on to prove that this is an onto map.

Let $|M, S, G| \in E_1(G)$. Take $x \in M$ and denote by $\overline{x}$ the map defined by sending $g \in G$ to $gx \in M$. Thus we define a subset, P, of $\underline{G}^*$ by $P = \{\overline{x}^*\omega : \omega \in S\}$.

As a consequence of Lemma 3.4 we have $\underline{N}_P = \underline{G}_x$. Thus N_P^0 is the component of the identity in the isotropy subgroup, G_x, at x. Since G_x is closed, it follows that N_P^0 is closed.

The form $\overline{x}^*\omega$ is Ad_g-invariant for all $g \in G_x$. Hence we have $G_x \subset G_P$, so that there exists $j \in J(P)$ such that $G_x = N_P^j$.

Thus, the map f defined by sending $gN_P^j \in G/N_P^j$ to $gx \in M$ is an equivariant diffeomorphism. We also have $f \circ p_j = \overline{x}$, where p_j is the canonical map from G onto G/N_P^j. Hence we have $p_j^* f^* \omega = \overline{x}^*\omega = p_j^*(\omega_P^j(\overline{x}^*\omega))$ for all $\omega \in S$. This proves that f is an isomorphism from $(G/N_P^j, P_1^j, G)$ onto (M, S, G). As a consequence, we have $\mu_1|P, N_P^j| = |G/N_P^j, P_1^j, G| = |M, S, G|$. Hence μ_1 is an onto map.

We have thus proved the following theorem which gives us a classification of $E_1(G)$.

Theorem 4.6. *The map defined by sending* $|P, N_P^j| \in K(G)$ *to* $|G/N_P^j, P_1^j, G| \in E_1(G)$ *is a bijection.*

Remark 4.7. The canonical map from G/N_P^0 onto G/N_P^j, p_{0j}, is a covering map and we have $p_{0j}^*(\omega_P^j(\delta)) = \omega_P^0(\delta)$ for all $\delta \in P$.

When G is simply connected G/N_P^0 is the universal covering space of each of the G/N_P^j.

5. Constructing homogeneous nondegenerate Pfaff systems

In order to explicitly construct the HNDPS for a given Lie group, G, one needs to know the subsets, P, of $\underline{G}^*$ such that N_P^0 is closed and, for these subsets, the subgroups N_P^j. To obtain this information by a direct use of the definition is not an easy task in general. In this section we reduce the problem to the search for some homomorphisms, which is in general much easier.

Let H and A be Lie groups and C a homomorphism from H onto A. Thus the transpose, ${}^t(\mathrm{d}C)$ of $\mathrm{d}C$ is a linear map from $\underline{A}^*$ into $\underline{H}^*$ whose image will be denoted by $\langle \mathrm{d}C \rangle$.

Let P be a subset of $\underline{G}^*$ containing some nonzero element and $i \in I(P)$. We consider homomorphisms, C, from $G^i{}_P$ onto connected abelian Lie groups such that $\langle \mathrm{d}C \rangle$ is the subspace of $\underline{G}_P^*$ generated by the restrictions to $\underline{G}_P$ of the elements of P. The set composed by these homomorphisms will be denoted $Hom^i{}_P$. Given a connected abelian Lie group, A, the subset of $Hom^i{}_P$ composed of the homomorphism whose image is A, will be denoted $Hom^i{}_P(A)$. If $C \in Hom^i{}_P(A)$, $\mathrm{d}C$ is onto, so that ${}^t(\mathrm{d}C)$ is injective thus giving an isomorphism from $\underline{A}^*$ onto the subspace generated by the restrictions to $\underline{G}_P$ of the elements of P.

Let $K_0'(G)$ be the set consists of the triples $(P, G^i{}_P, C)$ such that $C \in Hom^i{}_P$.

If $(P, G^i{}_P, C) \in K_0'(G)$, the kernel of $\mathrm{d}C$ is $\underline{N}_P$ and the kernel of C is closed. Hence N_P^0 is closed and there exists $j \in J(P)$ such that $Ker\, C = N_P^j$, i.e. $(P, Ker\, C) \in K_0(G)$. Thus, each element of $K_0'(G)$ gives rise to a HNDPS. The main objective of this section is to prove that each HNDPS can be obtained in this way. Some other results are proved in order to be used later.

Two elements of $K_0'(G)$, $(P, G^i{}_P, C)$ and $(P', G^{i'}{}_{P'}, C')$, are said to be equivalent if there exists $g \in G$ and an isomorphism f from $C(G^i{}_P)$ onto $C'(G^{i'}{}_{P'})$ such that $P' = Ad_g^*(P)$, $G^{i'}{}_{P'} = a_g(G^i{}_P)$ and $C' = f \circ C \circ a_{g^{-1}}$. This defines an equivalence relation in $K_0'(G)$. The equivalence class of $(P, G^i{}_P, C)$ will be denoted by $|P, G^i{}_P, C|$ and the quotient set by $K'(G)$.

Lemma 5.1. *Let* $(P, G^i{}_P, C), (P', G^{i'}{}_{P'}, C') \in K_0'(G)$. *Then we have* $|P, Ker\, C| = |P', Ker\, C'|$ *if and only if* $|P, G^i{}_P, C| = |P', G^{i'}{}_{P'}, C'|$.

Proof. Let $j \in J(P)$, $j' \in J(P')$ be such that $Ker\, C = N_P^j$, $Ker\, C' = N_{P'}^{j'}$, and assume that $|P, N_P^j| = |P', N_{P'}^{j'}|$. Then there exists $g \in G$ such that $P' = Ad_g^*(P)$ and $N_{P'}^{j'} = a_g(N_P^j)$.

Let $\{a_1, \dots, a_r\}$ be a basis of $\underline{A}$, where A is the image of C. Let $Y_1 \cdots Y_r \in \underline{G}_P$ be such that $dC(Y_k) = a_k$ for all $k = 1, \dots, r$. Thus, for all $h \in G^i_P$ there exists $t_1 \cdots t_r$ such that $C(h) = Exp(t_1 a_1) \cdots Exp(t_r a_r) = Exp(dC(t_1 Y_1)) \cdots Exp(dC(t_r Y_r)) = C(Exp(t_1 Y_1) \cdots Exp(t_r Y_r))$. As a consequence, all the elements of G^i_P can be written in the form $Exp(t_1 Y_1) Exp(t_1 Y_1) \cdots Exp(t_r Y_r) n$, where $t_1 \dots t_r \in \mathbb{R}$ and $n \in N^j_P$. But $g\, Exp(t_1 Y_1) \cdots Exp(t_r Y_r) n g^{-1} = Exp(t_1\, Ad_g(Y_1)) \cdots Exp(t_r\, Ad_g(Y_r)) g n g^{-1}$, and, because of Lemma 4.2, we have $\underline{G}_{P'} = Ad_g(\underline{G})$, so that $Exp(t_k\, Ad_g(Y_k)) \in G^0_{P'} \subset G^{i'}_{P'}$. Thus $a_g(G^i_P) \subset G^{i'}_{P'}$.

In a similar way we obtain $a_{g^{-1}}(G^{i'}_{P'}) \subset G^i_{P'}$, so that $G^{i'}_{P'} = a_g(G^i_P)$.

Now we define a map, $f_{1'}$ from G^i_P / N^j_P onto $G^{i'}_{P'} / N^{j'}_{P'}$ by means of $f_1(hN^j_P) = ghg^{-1} N^{j'}_{P'}$. This is a well defined Lie group isomorphism, whose inverse can be defined in a similar way. We have $C' = f_3 \circ f_1 \circ f_2^{-1} \circ C \circ a_{g^{-1}}$, where f_2(resp. f_3) is the isomorphism from G^i_P / N^j_P(resp. $G^{i'}_{P'} / N^{j'}_{P'}$) onto the image of C(resp. C') canonically defined by C (resp. C').

The "if" part is immediate. □

As a consequence of Lemma 5.1 we see that the map, μ_2, defined by sending $|P, G^i_P, C| \in K'(G)$ to $|P, Ker\, C| \in K(G)$ is a well defined injective map. We shall now prove that this is an onto map.

Lemma 5.2. *Let $(P, N^j_P) \in K_0(G)$. Then, for all $Y \in \underline{G}_P$, there exists a unique differentiable vector field on G/N^j_P, $Z^j_P(Y)$, such that $(p_j)_* Y = Z^j_P(Y)$. Moreover, for all $\sigma \in P$, we have $\omega^j_P(\sigma)(Z^j_P(Y)) = \sigma(Y)$, and $i(Z^j_P(Y))\, d(\omega^j_P(\sigma)) = 0$.*

Proof. Let $g \in G$, $n \in N^j_P$, $Y \in \underline{G}_P$ and $X \in \underline{G}$. Then we have

$$\sigma(Y) = \sigma_{gn}(Y_{gn}) = (\omega^j_P(\sigma))_{gN^j_P}((p_j)_* Y_{gn}),$$

$$0 = (i(Y)\, d\sigma)(X) = (d(\omega^j_P(\sigma)))_{gN^j_P}((p_j)_* Y_{gn}, (p_j)_* X_{gn}).$$

Since P^j_1 is a nondegenerate 1-system , it follows that the above relations define uniquely $(p_j)_* Y_{gn}$. Consequently, $(p_j)_* Y_{gn}$ does not depend on n.

Thus we define $Z^j_P(Y)$ by means of $(Z^j_P(Y))_{gN^j_P} = (p_j)_* Y_g$. Since p_j admits local cross sections everywhere, $Z^j_P(Y)$ is differentiable. □

Let $(P, N^j_P) \in K_0(G)$ and let D be the connected component of $N^j_P \in G_P / N^j_P$ in G_P / N^j_P. The stabilizer of D, by the canonical action of G_P on G_P / N^j_P, is a subgroup of G_P which contains G^0_P. Thus, there exists $i(j) \in I(P)$ such that $G^{i(j)}_P$ is the stabilizer of D.

Remark 5.3. If P consists of a single 1-form, σ, and $\omega^j_P(\sigma)$ is a contact form , there exists $Y \in \underline{G}$ such that $\sigma(Y) = 1$ and $Z^j_P(Y)$ is the canonical vector field associated to $\omega^j_P(\sigma)$ in the sense of Example 3.2. If the differential of $\omega^j_P(\sigma)$ is symplectic, $\underline{N}_P = \underline{G}_P$, and all the $Z^j_P(Y)$ are 0.

Lemma 5.4. *The open submanifold D is $G^{i(j)}_P/N^j_P$ provided with its homogeneous differentiable structure.*

Proof. The action of G_P on G_P/N^j_P gives us a differentiable action of $G^{i(j)}_P$ on D. It is enough to prove that this action is transitive.

Given two arbitrary points of D, there exist $g \in G_P$ which transform the first of these points into the second. Since $g(D)$ must be a connected component of G_P/N^j_P, and $g(D) \cap D \neq \emptyset$ we have $g(D) = D$ so that $g \in G^{i(j)}_P$. □

Lemma 5.5. *N^j_P is a normal subgroup of $G^{i(j)}_P$ and $G^{i(j)}_P = G^0_P N^j_P$.*

Proof. Since $G^{i(j)}_P/N^j_P$ is connected, it follows that the component of the identity in $G^{i(j)}_P$ acts transitively on it. Thus $G^{i(j)}_P = G^0_P N^j_P$.

Now let $n \in N^j_P$, $g \in G^{i(j)}_P$. In order to prove that $g^{-1}ng \in N^j_P$ it suffices to prove it when $g_0 \in G^0_P$. Thus, we only need to prove that $(Exp\, X)^{-1} n\, Exp\, X \in N^j_P$ for all $X \in \underline{G}_P$. But $n\, Exp\, tX$ is the integral curve of X having n as initial value, so that $n(Exp\, tX)N^j_P$ is the integral curve of $Z^j_P(X)$ having N^j_P as initial value. The same holds for all $n \in N^j_P$. Hence we have $n(Exp\, tX)N^j_P = (Exp\, tX)N^j_P$ for all $t \in \mathbb{R}$, and the result follows. □

Thus $G^{i(j)}_P/N^j_P$ is a connected Lie group whose Lie algebra is isomorphic to $\underline{G}_P/\underline{N}_P$. Since $\underline{G}_P/\underline{N}_P$ is an abelian Lie algebra, it follows that $G^{i(j)}_P/N^j_P$ is an abelian group.

In the following, the Lie algebra of $G^{i(j)}_P/N^j_P$ will be identified to $\underline{G}_P/\underline{N}_P$ by means of the canonical isomorphism. Thus, for all $X \in \underline{G}_P$ we have $Exp(X + \underline{N}_P) = (Exp\, X)N^j_P$.

Lemma 5.6. *The map, q, from P into the dual, $(\underline{G}_P/\underline{N}_P)^*$, of $\underline{G}_P/\underline{N}_P$ defined by*

$$q(\sigma)(X + \underline{N}_S) = \sigma(X)$$

for all $\sigma \in P$, $X \in \underline{G}_P$ is a well defined map, whose image contains a basis of the dual of $\underline{G}_P/\underline{N}_P$.

Proof. For all $\sigma \in P$, $q(\sigma)$ is a well defined linear functional on $\underline{G}_P/\underline{N}_P$.

Now let $\sigma^1, \ldots, \sigma^r \in P$ be such that $\{\sigma^1|_{\underline{G}_P}, \ldots, \sigma^r|_{\underline{N}_P}\}$ is a maximal linearly independent subset of $\{\sigma|_{\underline{G}_P} : \sigma \in P\}$. Then we have $dim(\underline{G}_P/\underline{N}_P) = r$ and $q(\sigma^1), \ldots, q(\sigma^r)$ is linearly independent. □

Let C^j_P be the canonical homomorphism from $G^{i(j)}_P$ onto $G^{i(j)}_P/N^j_P$. Thus we have $dC^j_P(X) = X + \underline{N}_P$ for all $X \in G_P$. Hence, the transpose of dC^j_P, ${}^t(dC^j_P)$, is such that ${}^t(dC^j_P)(q(\sigma)) = \sigma|_{\underline{G}_P}$, for all $\sigma \in P$. Then $(P, G^{i(j)}_P, C^j_P) \in K'_0(G)$ and

$\mu_2(|P, G_P^{i(j)}, C_P^j|) = |P, N_P^j|$. Thus we have the following theorem.

Theorem 5.7. *The map defined by sending $|P, G^i_P, C| \in K'(G)$ to $|P, Ker\, C| \in K(G)$ is a bijection.*

This gives us an alternative form of the classification of $E_1(G)$.

We quote here, for future reference, the following lemmas.

Lemma 5.8. *Let $j \in J(P)$ and $Y_1, \ldots, Y_r \in \underline{G}_P$ a basis of a supplementary of $\underline{N}_P$. Thus*

$$G_P^{i(j)} = \{(Exp\, t^1 Y_1) \cdots (Exp\, t^r Y_r) n : n \in N_P^j, t^1, \ldots, t^r \in \mathbb{R}\}.$$

To prove this lemma one can proceed as in the proof of Lemma 5.1 with $C = C_P^j$, $A = G_P^{i(j)}/N_P^j$, $a_k = Y_k + \underline{N}_P$, $i = i(j)$.

Lemma 5.9. *Let $j \in J(P), i \in I(P)$. We have $i = i(j)$ if and only if $N_P^j \subset G^i_P$ and G^i_P/N_P^j is a connected subset of G_P/N_P^j.*

Proof. The "only if" part is obvious. If $N_P^j \subset G^i_P$ and G^i_P/N_P^j is connected, thus G_P^0 acts transitively on G^i_P/N_P^j. It follows that for all $g \in G^i_P$ there exists $h \in G_P^0$ such that $gN_P^j = hN_P^j$. Hence $G^i_P \subset G_P^0 N_P^j = G^{i(j)}$. Since $G_P^0 N_P^j \subset G^i_P$ the result follows. □

6. Multihamiltonian spaces

If $(P, G^i_P, C) \in K_0'(G)$, and P is completely regular in a sense to be explained in Section 7, $(G/Ker\, C)(G/G^i_P, C(G^i_P))$ is a principal fibre bundle with abelian structural group. There is a family of connections on this bundle canonically associated with P. The differentials of the elements of P are projectable on the base space and define a 2-system on it, having some special properties. This 2-system is related to the curvature forms of the connections. Thus, the HNDPS gives rise to principal fibre bundles with connections and the base spaces of these bundles have a special geometrical structure. The study of these spaces is the objective of this section.

A multihamiltonian G-space is a triple (B, D, G) where B is a connected differentiable manifold, G is a connected Lie group and D is a set whose elements are pairs (Ω, λ) where Ω is a differential 2-form on B and λ is a map from $\underline{G}$ into $C^\infty(B)$ such that:

(1) (B, D_1, G) is a G-homogeneous nondegenerate 2-system, where D_1 is the set consisting of the first components of the elements of D.

(2) $i(X_B)\Omega = \mathrm{d}(\lambda(X))$ for all $(\Omega, \lambda) \in D$, $X \in \underline{G}$, where X_B is the infinitesimal generator of the action of G on B associated with X.

(3) λ is $\mathbb{R}$-linear for all $(\Omega, \lambda) \in D$.

(4) $\lambda([X, Y]) = \Omega(Y_B, X_B)$ for all $(\Omega, \lambda) \in D$, $X, Y \in \underline{G}$.

Lemma 6.1. *Let (B, D, G) be a multihamiltonian G-space and $(\Omega, \lambda) \in D$. Then Ω is closed.*

Proof. Since Ω is G-invariant, it follows that $L(X_B)\Omega = 0$ for all $X \in \underline{G}$. Thus condition (2) implies $i(X_B)\, \mathrm{d}\Omega = 0$. Since the action is transitive, it follows that $\mathrm{d}\Omega = 0$. □

Let (B, D, G) be a multihamiltonian G-space where D is composed by a single element (Ω, λ). Then Ω is a nondegenerated closed 2-form, i.e. a symplectic form. Conditions (3) and (4) tell us that λ is a homomorphism of Lie algebras when $C^\infty(B)$ is provided with the Poisson bracket associated to Ω. Hence (B, Ω, λ, G) gives us a Hamiltonian space in the sense of Kostant [7].

This section will be devoted to the classification of the multihamiltonian G-spaces (up to isomorphism). This classification generalizes some of the well known Kostant results on Hamiltonian G-spaces.

Let (B, D, G) and (B', D', G) be multihamiltonian G-spaces. An isomorphism from (B, D, G) onto (B', D', G) is a pair (f, h), where f is a G-equivariant diffeomorphism from B onto B' and h is a bijective map from D onto D' such that $(h(\lambda))(X) \circ f = \lambda(X)$ for all $(\Omega, \lambda) \in D$, $X \in \underline{G}$, where $h(\lambda)$ is the second component in $h(\Omega, \lambda)$.

Let (f, h) be an isomorphism as above,$(\Omega, \lambda) \in D$ and let us denote by $h(\Omega)$ the first component in $h(\Omega, \lambda)$. Then we have $f^*(h(\Omega)) = \Omega$. In fact, for all $X, Y \in \underline{G}$, we have

$$[f^*(h(\Omega))](X_B, Y_B) = h(\Omega)(X_{B'}, Y_{B'}) \circ f = [h(\lambda)([Y, X])] \circ f$$
$$= \lambda([Y, X]) = \Omega(X_B, Y_B).$$

Isomorphy defines an equivalence relation in the set of multihamiltonian G-spaces. The equivalence class of (B, D, G) will be denoted by $|B, D, G|$ and the quotient set by $Mham(G)$.

Let $Mh_0(G)$ be the set composed by the pairs (P, G^i_P) where P is a subset of $\underline{G}^*$ such that $\underline{G}_P \neq \underline{G}$ and $i \in I(P)$. Two elements, (P, G^i_P) and $(Q, G^{i'}_Q)$, of $Mh_0(G)$ are said to be equivalent if there exists $g \in G$ such that $Q = Ad^*_g(P)$ and $G^{i'}_Q = a_g(G^i_P)$. The equivalence class of (P, G^i_P) will be denoted by $|P, G^i_P|$ and the quotient set by $Mh(G)$.

We shall give a classification of multihamiltonian G-spaces (up to equivalence) by giving a bijective map from $Mh(G)$ onto $Mham(G)$.

Let $(P, G^i_P) \in Mh_0(G)$ and let p_i be the canonical map from G onto G/G^i_P. The definitions of $\underline{G}_P$ and G^i_P entail the following lemma.

Lemma 6.2. *For all $\sigma \in P$ there exists a unique G-invariant 2-form $\Omega^i_P(\sigma)$ on G/G^i_P such that $p_i^*(\Omega^i_P(\sigma)) = \mathrm{d}\sigma$.*

To each $(\sigma, X) \in P \times \underline{G}$ we associate a function, $\lambda^i_P(\sigma, X)$, on G/G^i_P by means of $[\lambda^i_P(\sigma, X)](gG^i_P) = Ad^*_g\sigma(X)$.

The function $\lambda^i_P(\sigma, X)$ is C^∞ since $\lambda^i_P(\sigma, X) \circ p_i \in C^\infty(G)$ and p_i admits local cross section everywhere on G/G^i_P.

Let $\lambda^i_P(\sigma)$ be the map defined by sending each $X \in \underline{G}$ to $\lambda^i_P(\sigma, X)$. It is a $\mathbb{R}$-linear map. Then we have the following lemma.

Lemma 6.3. $(G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)) : \sigma \in P\}, G)$ *is a multihamiltonian* G*-space.*

Proof. In order to show that $(G/G^i_P, \{\Omega^i_P(\sigma) : \sigma \in P\}, G)$ is a homogeneous nondegenerate 2-system,we only need to prove that $\{\Omega^i_P(\sigma) : \sigma \in P\}$ is nondegenerate. Since each $\Omega^i_P(\sigma)$ is invariant and the action is transitive we only need to prove that this condition holds at a point.

We identify the tangent space to G/G^i_P at G^i_P with $\underline{G}/\underline{G}_P$ in the canonical way. Let $X \in \underline{G}$ be such that $i(X + \underline{G}_P)\Omega^i_P(\sigma) = 0$ for all $\sigma \in P$. Then we have $i(X)\,\mathrm{d}\sigma = 0$ for all $\sigma \in P$, i.e. $X + \underline{G}_P = \underline{G}_P$.

Let us denote by X^i_P the infinitesimal generator of the action of G on G/G^i_P associated to $X \in \underline{G}$. Then, for all $X, Y \in \underline{G}$ and $\sigma \in P$, we have

$$
\begin{aligned}
&[\mathrm{d}(\lambda^i_P(\sigma, X))(Y^i_P)](gG^i_P)\\
&= \mathrm{d}/\mathrm{d}t|_{t=0}(\lambda^i_P(\sigma, X))(Exp(-tY)gG^i_P)\\
&= \mathrm{d}/\mathrm{d}t|_{t=0}(Ad^*_g\sigma)(Ad_{Exp\,tY}X) = (Ad^*_g\sigma)([Y, X])\\
&= \mathrm{d}\sigma(Ad_{g^{-1}}X, Ad_{g^{-1}}Y) = [\Omega^i_P(\sigma)(-X^i_P, -Y^i_P)](gG^i_P),
\end{aligned}
$$

since $(p_i)_*(Ad_{g^{-1}}X)_g = -(X^i_P)_{gG^i_P}$. Hence we have $i(X^i_P)\Omega^i_P(\sigma) = \mathrm{d}(\lambda^i_P(\sigma, X))$ and $[\lambda^i_P(\sigma, [X, Y])](gG^i_P) = (Ad^*_g\sigma)([X, Y]) = [\Omega^i_P(\sigma)(Y^i_P, X^i_P)](gG^i_P)$. □

Lemma 6.4. *Let* $(P, G^i_P), (Q, G^j_Q) \in Mh_0(G)$. *Then we have* $|P, G^i_P| = |Q, G^i_Q|$ *if and only if* $|G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)) : \sigma \in P\}, G| = |G/G^j_Q, \{(\Omega^j_Q(\omega), \lambda^j_Q(\omega)) : \omega \in Q\}, G|$.

Proof. We first prove the "only if" part. Let $g \in G$ be such that $Q = Ad^*_g(P)$ and $G^j_Q = a_g(G^i_P)$. Thus we define a map, b_g, by sending $hG^i_P \in G/G^i_P$ to $hg^{-1}G^j_Q \in G/G^j_Q$. This is a well defined map. Since we have $q_j \circ R_{g^{-1}} = b_g \circ p_i$, and p_i admits local cross sections everywhere on its image, it follows that b_g is differentiable. Actually b_g is a diffeomorphism whose inverse can be defined in a similar way. Also we see that b_g is G-equivariant.

Now we shall prove that (b_g, A) is an isomorphism of multihamiltonian G-spaces, where A maps $(\Omega^i_P(\sigma), \lambda^i_P(\sigma))$ to $(\Omega^j_Q(Ad^*_g\sigma), \lambda^j_Q(Ad^*_g\sigma))$. In fact, for all $X \in \underline{G}$, $h \in G$ and $\sigma \in P$, we have

$$
[\lambda^j_Q(Ad^*_g\sigma, X)] \circ b_g(hG^i_P) = Ad^*_{hg^{-1}}Ad^*_g\sigma(X) = Ad^*_h\sigma(X) = [\lambda^i_P(\sigma, X)](hG^i_P).
$$

Conversely, let (f, h) be an isomorphism from $(G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)) : \sigma \in P\}, G)$ onto $(G/G^j_Q, \{(\Omega^j_Q(\omega), \lambda^j_Q(\omega)) : \omega \in Q\}, G)$.

Let $g \in G$ be such that $f(G^i_P) = gG^j_Q$. If $\sigma \in P$, there exists $\sigma' \in Q$ such that $h(\Omega^i_P(\sigma), \lambda^i_P(\sigma)) = (\Omega^j_P(\sigma'), \lambda^j_P(\sigma'))$. Then we have

$$
\sigma(X) = [\lambda^i_P(\sigma, X)](G^i_P) = [\lambda^j_Q(\sigma', X)] \circ f(G^i_P) = Ad^*_g\sigma'(X)
$$

for all $X \in \underline{G}$, so that $\sigma = Ad^*_g\sigma'$. It follows that $Ad^*_{g^{-1}}(P) \subset Q$. Since $f^{-1}(G^j_P) = g^{-1}G^i_P$ (because of the equivariance) we have $Ad^*_g(Q) \subset P$. Thus $P = Ad^*_g(Q)$.

Since f is equivariant, it follows that the isotropy subgroup at gG^j_Q (i.e. $gG^j_Q g^{-1}$) must coincide with the isotropy subgroup at G^i_P (i.e. G^i_P). Hence we have $G^i_P = a_g(G^j_Q)$. □

Theorem 6.5. *The map from $Mh(G)$ into $Mham(G)$ defined by sending $|P, G^i_P|$ to $|G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)\colon \sigma \in P\}, G|$ is a bijection.*

Proof. Because of the preceding lemma this is a well defined injective map which will be denoted by β. We only need to prove that β is onto. Let $(B, \{(\Omega_a, \lambda_a)\colon a \in A\}, G)$ be a multihamiltonian G-space, where A is an index set. For each $a \in A$ we define a map, $\underline{\lambda}_a$ from B into $\underline{G}^*$ by means of $[\underline{\lambda}_a(b)](X) = [\lambda_a(X)](b)$, for all $b \in B$, $X \in \underline{G}$. The map $\underline{\lambda}_a$ is differentiable. In fact, let $X_1, \dots, X_n$ be a basis of $\underline{G}$, which provides us with a coordinate system of $\underline{G}^*$ by identifying $\underline{G}^{**}$ with $\underline{G}$. Each $X_i \circ \underline{\lambda}_a$ is C^∞ since it coincides with $\lambda_a(X_i)$.

For all $b \in B$, $X, Y \in \underline{G}$, we have

$$\begin{aligned}
&\mathrm{d}/\mathrm{d}t|_{t=0}[\underline{\lambda}_a((Exp(-tY))b)](X)\\
&= \mathrm{d}/\mathrm{d}t|_{t=0}[\lambda_a(X)]((Exp(-tY))b) = (Y_B)_b(\lambda_a(X))\\
&= [\Omega_a(X_B, Y_B)](b) = [\lambda_a([Y, X])](b) = [\underline{\lambda}_a(b)]([Y, X])\\
&= [\underline{\lambda}_a(b)](ad_Y(X)) = \mathrm{d}/\mathrm{d}t|_{t=0}[\underline{\lambda}_a(b)]((Exp\, t ad_Y)(X))\\
&= \mathrm{d}/\mathrm{d}t|_{t=0}[\underline{\lambda}_a(b)](Ad_{Exp\, tY}(X)) = \mathrm{d}/\mathrm{d}t|_{t=0}(Ad^*_{Exp(-tY)}[\underline{\lambda}_a(b)])(X).
\end{aligned}$$

But this implies that $(\underline{\lambda}_a)_* Y_B = Y_{\underline{G}^*}$, where $Y_{\underline{G}^*}$ is the infinitesimal generator of the coadjoint action associated with Y. Since G is connected, it follows that $\underline{\lambda}_a$ is equivariant.

Now fix $b \in B$ and denote $P = \{\underline{\lambda}_a(b)\colon a \in A\}$. Since all $\underline{\lambda}_a$ are equivariant, it follows that the isotropy subgroup at b, G_b, is contained in G_P.

On the other hand we have

$$(i(Y_B)_b\Omega_a)((X_B)_b) = (\lambda_a([X, Y]))(b) = [\underline{\lambda}_a(b)]([X, Y]) = [L(Y)(\underline{\lambda}_a(b))](X)$$

for all $a \in A$, $X, Y \in \underline{G}$. Since $\{\Omega_a\colon a \in A\}$ is nondegenerated and each Ω_a is closed, it follows that we have $(Y_B)_b = 0$ if and only if $L(Y)(\underline{\lambda}_a(b)) = 0$ for all $a \in A$. Hence the Lie algebra of G_b is $\underline{G}_P$. Thus there exists $i \in I(P)$ such that $G_b = G^i_P$.

The proof will be finished when we prove that $(B, \{(\Omega_a, \lambda_a)\colon a \in A\}, G)$ is isomorphic to $(G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma))\colon \sigma \in P\}, G)$.

Let h be the map from $\{(\Omega_a, \lambda_a)\colon a \in A\}$ into $\{(\Omega^i_P(\sigma), \lambda^i_P(\sigma))\colon \sigma \in P\}$ defined by $h(\Omega_a, \lambda_a) = (\Omega^i_P(\underline{\lambda}_a(b)), \lambda^i_P(\underline{\lambda}_a(b)))$. This map is obviously onto. Later on we shall prove that h is also injective.

Let $\overline{b}$ be the equivariant diffeomorphism from G/G_b onto B defined by sending gG_b to gb. Then, for all $a \in A$, $X \in \underline{G}$, we have

$$\lambda_a(X) \circ \overline{b}(gG^i_P) = [\underline{\lambda}_a(gb)](X) = (Ad^*_g(\underline{\lambda}_a(b)))(X) = [\lambda^i_P(\underline{\lambda}_a(b), X)](gG^i_P)$$

so that

$$\lambda_a(X) \circ \bar{b} = [\lambda^i_P(\underline{\lambda}_a(b))](X). \tag{1}$$

As a consequence of (1) we see that the relation $h(\Omega_a, \lambda_a) = h(\Omega_{a'}, \lambda_{a'})$ implies $\lambda_a(X) \circ \bar{b} = \lambda_{a'}(X) \circ \bar{b}$. Then we have $\lambda_a = \lambda_{a'}$ and this relation implies easily that $\Omega_a = \Omega_{a'}$. We thus see that h is injective.

The pair $(\bar{b}, h^{-1})$ is an isomorphism from $(G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)): \sigma \in P\}, G)$ onto $(B, \{(\Omega_a, \lambda_a): a \in A\}, G)$ because of (1). □

Remark 6.6. Let P be such that $\underline{G}_P \neq \underline{G}$. There exist $k \in I(P)$ such that $G_P = G^k_P$. Let us denote $\Omega_P(\sigma) = \Omega^k_P(\sigma)$, $\lambda_P(\sigma) = \lambda^k_P(\sigma)$ for all $\sigma \in P$. Thus $(G/G_P, \{(\Omega_P(\sigma), \lambda_P(\sigma)): \sigma \in P\}, G)$ is a multihamiltonian G-space.

The canonical map from G/G^i_P onto G/G_P, p^i_P, is a covering map. We have $\Omega^i_P(\sigma) = (p^i_P)^*\Omega_P(\sigma)$ and $\lambda^i_P(\sigma, X) = \lambda_P(\sigma, X) \circ p^i_P$ for all $\sigma \in P$, $X \in \underline{G}$.
The canonical map from G/G^0_P onto G/G^i_P, p^0_i, is a covering map and we have $\Omega^0_P(\sigma) = (p^0_i)^*\Omega^i_P(\sigma)$, $\lambda^0_P(\sigma, X) = \lambda^i_P(\sigma, X) \circ p^0_i$ for all $\sigma \in P$, $X \in \underline{G}$.

When G is simply connected, G/G^0_P is the universal covering space of all of the G/G^i_P and any covering space of G/G_P is diffeomorphic to a G/G^i_P.

7. Fibre bundles arising from homogeneous nondegenerate 1-systems

Let P be a subset of $\underline{G}^*$ containing some nonzero element.

P is said to be *regular* if N^0_P is closed and $\underline{G}_P \neq \underline{N}_P$. Otherwise P is said to be *singular*. P is said to be *algebraically singular* when $\underline{G}_P = \underline{N}_P$. When N^0_P is not closed, then we say that P is *topologically singular*. Note that, if P is algebraically singular, then we have $N^0_P = G^0_P$, so that P cannot be topologically singular. If P is regular and $\underline{G}_P \neq \underline{G}$, then we say that P is *completely regular*.

The adjectives regular, topologically singular, algebraically singular or completely regular are also applied to elements (P, N^j_P) of $K_0(G)$ or (P, G^i_P, C) of $K'_0(G)$ when P is regular, etc. A multihamiltonian space (resp. homogeneous nondegenerate 1-system) is said to be regular, etc. if it is equivalent to one of the forms $(G/G^i_P, \{(\Omega^i_P(\sigma), \lambda^i_P(\sigma)): \sigma \in P\}, G)$ (resp. $|G/N^j_P, P^j_1, G|$) where P is regular, etc. The adjectives are also applied to the equivalence classes of these objects.

Lemma 7.1. *Let $P \in \underline{G}^*$ be such that $P \neq \{0\}$. Then*
(a) *P is regular if and only if Hom^0_P contains some nontrivial homomorphism.*
(b) *P is algebraically singular if and only if $Hom^0_P = \{\tau\}$, where τ is the trivial onto homomorphism (from G^0_P onto $\{e\}$).*
(c) *P is topologically singular if and only if Hom^0_P is empty.*

Proof. Let P be regular. We have $G^{i(0)}_P = G^0_P$. Thus $C^{i(0)}_P \in Hom^0_P(G^0_P/N^0_P)$ and $C^{i(0)}_P$ is not trivial.

Conversely, let $C \in Hom^0_P(A)$ be nontrivial. Thus N^0_P is closed and $dim(\underline{G}_P/\underline{N}_P) = dim(\underline{A}) \neq 0$. Hence σ is regular.

The proof of (b) and (c) is left to the reader. □

Let $(P, G^i{}_P, C) \in K'_0(G)$, $N^j_P = Ker\, C$, where P is completely regular. Thus $i = i(j)$ (cf. Lemma 5.9).

We know that $G/N^j_P(G/G^i{}_P, G^i{}_P/N^j_P)$ is a principal fibre bundle with the canonical map, p_{ji}, from G/N^j_P onto $G/G^i{}_P$ as bundle projection.

We have two actions on G/N^j_P. Firstly, the canonical action on the left of G on G/N^j_P: $g \cdot (hN^j_P) = (gh)N^j_P$. On the other hand, we have the action on the right of the structural group, $G^i{}_P/N^j_P$, on G/N^j_P: $(gN^j_P) \cdot (hN^j_P) = (hg)N^j_P$ for all $g \in G^i{}_P$, $h \in G$. This action on the right will be referred to as *bundle action*.

The Lie algebra of $G^i{}_P/N^j_P$ will be identified in the canonical way with $\underline{G}_P/\underline{N}_P$.

Lemma 7.2. *The infinitesimal generator of the bundle action associated with* $Y + \underline{N}_P$, $Y \in \underline{G}_P$ *is* $Z^j_P(Y)$.

Proof. The value at hN^j_P of the infinitesimal generator associated with $Y + \underline{N}_P$, since the action is on the right, is the tangent vector to

$$Exp\, t(Y + \underline{N}_P) \cdot (hN^j_P) = h\, Exp\, tY N^j_P = p_j \cdot R_{Exp\, tY}(h).$$

But the tangent vector to this curve is $p_{j*}(Y_h) = (Z^j_P(Y))(p_j(h))$. □

Remark 7.3. The canonical action and the bundle action commute each other. Therefore, the vector fields $Z^j_P(Y)$ remain invariant under the canonical action and the infinitesimal generators of the canonical action remain invariant under the bundle action.

Let us denote by X^j_P the infinitesimal generator of the canonical action associated with $X \in G$. Let φ be the diffeomorphism of G/N^j_P associated, by the bundle action, with $gN^j_P \in G^i{}_P/N^j_P$.

We have $\varphi \circ p_j = p_j \circ R_g$, so that

$$p^*_j(\varphi^* \omega^j_P(\sigma)) = R^*_g(p^*_j \omega^j_P(\sigma)) = R^*_g \sigma = Ad^*_g \sigma = \sigma = p^*_j(\omega^j_P(\sigma)).$$

We have thus proved the following lemma.

Lemma 7.4. $\omega^j_P(\sigma)$ *is invariant for the bundle action for all* $\sigma \in P$.

Let $\underline{P}$ be the subset of $(\underline{G}_P)^*$ composed of the restrictions to $\underline{G}_P$ of the elements of P and let $\langle \underline{P} \rangle$ be the subspace of $(\underline{G}_P)^*$ generated by $\underline{P}$. The dimension of $\langle \underline{P} \rangle$ will be denoted by $r(P)$ and we have $r(P) = dim\, \underline{G}_P - dim\, \underline{N}_P$. Also $r(P)$ is the cardinal of each maximal linearly independent subset of $\underline{P}$.

Let $\underline{G}^{\perp}_P$ be the set consisting of $\sigma \in \underline{G}^*$ such that $\underline{\sigma} = 0$, where $\underline{\sigma}$ means the restriction of σ to $\underline{G}_P$.

We denote by $S(P)$ the set consisting of $r(P)$-dimensional subspaces of $\underline{G}^*$ such that $S \cap P$ contains a basis of S and $S \cap \underline{G}_P^{\perp} = \{0\}$.

The elements of $S(P)$ can be obtained as follows. Let $\sigma^1, \ldots, \sigma^{r(P)} \in P$ be such that $\{\underline{\sigma}^1, \ldots, \underline{\sigma}^{r(P)}\}$ is a maximal linearly independent subset of $\underline{P}$. Then the subspace of $\underline{G}^*$ spanned by $\{\sigma^1, \ldots, \sigma^r\}$, $\langle\sigma^1, \ldots, \sigma^r\rangle$, is an element of $S(P)$.

Let $S \in S(P)$ and let $\sigma^1, \ldots, \sigma^{r(P)} \in P$ be a basis of S. We define an isomorphism, q_S, from S onto $(\underline{G}_P/\underline{N}_P)^*$ by means of $(q_S(\sigma))(X + \underline{N}_P) = \sigma(X)$ for all $\sigma \in S$, $X \in \underline{G}_P$. On the other hand, we know that for all $\sigma \in S$ there exists a unique $\omega^j_P(\sigma) \in \Omega^1(G/N^j_P)$ such that $\sigma = p_j^*(\omega^j_P(\sigma))$ and a unique $\Omega^i_P(\sigma) \in \Omega^2(G/G^i_P)$ such that $p_i^*\Omega^i_P(\sigma) = \mathrm{d}\sigma$. Thus we have the following lemma.

Lemma 7.5. *Let $g \in G$, $v \in T_{gN^j_P}(G/N^j_P)$. Thus there exists a unique element of $\underline{G}_P/\underline{N}_P$, which we shall denote by $\tilde{\omega}^j_P(S)_{gN^j_P}(v)$ such that*

$$q_S(\sigma)(\tilde{\omega}^j_P(S)_{gN^j_P} \cdot (v)) = \omega^j_P(\sigma)_{gN^j_P}(v)$$

for all $\sigma \in S$.

Proof. Since $\underline{\sigma}^1, \ldots, \underline{\sigma}^{r(P)}$ are linearly independent, there exist $y_1, \ldots, y_{r(P)} \in \underline{G}_P$ such that $\sigma^k(y_i) = \delta^k_i$ for all $i, k = 1, \ldots, r(P)$. Thus $\{q_S(\sigma^k)\}$ is the dual basis of $\{y_k + \underline{N}_P\}$.

If $\tilde{\omega}^j_P(S)_{gN^j_P}(v)$ exists, we must have

$$\tilde{\omega}^j_P(S)_{gN^j_P} \cdot (v) = \sum_{k=1}^{r(P)} (\omega^j_P(\sigma^k)_{gN^j_P} \cdot (v))(y_k + \underline{N}_P),$$

which proves uniqueness. To prove existence, it suffices to define $\tilde{\omega}^j_P(S)_{gN^j_P}(v)$ by the above relation. □

This lemma enables us to consider the differential 1-form $\tilde{\omega}^j_P(S)$, on G/N^j_P whose value at gN^j_N sends $v \in T_{gN^j_P}(G/N^j_P)$ to $\tilde{\omega}^j_P(S)_{gN^j_P}(v)$.

With the usual identifications and the notation of the proof of Lemma 7.5 we can write

$$\tilde{\omega}^j_P(S) = \sum_{k=1}^{r(P)} \omega^j_P(\sigma^k) \otimes (y_k + \underline{N}_P).$$

As a consequence of Lemmas 7.4, 7.2 and 5.2, we see that this form is a connection form on $G/N^j_P(G/G^i_P, G^i_P/N^j_P)$.

Since the structural group is abelian , the curvature form is the differential of the connection and will be identified to its projection on the base space. Thus the curvature form is given by

$$\tilde{\Omega}^i_P(S) = \sum_{k=1}^{r(P)} \Omega^i_P(\sigma^k) \otimes (y_k + \underline{N}_P).$$

In fact, for all $k = 1, \dots, r(P)$, we have

$$p_j^*(p_{ji}^* \Omega^i{}_P(\sigma^k)) = p_i^* \Omega^i{}_P(\sigma^k) = \mathrm{d}\sigma^k = p_j^* \mathrm{d}(\omega^j{}_P(\sigma^k))$$

so that $p_{ij}^* \Omega^i{}_P(\sigma^k) = \mathrm{d}(\omega^j{}_P(\sigma^k))$.

Proposition 7.6. *Let $S, S' \in S(P)$. Then there exists on $G/G^i{}_P$ an invariant 1-form with values in $\underline{G}_P/\underline{N}_P$, α, such that*

$$\tilde{\omega}^j{}_P(S') = \tilde{\omega}^j{}_P(S) + p_{ji}^* \alpha, \qquad \tilde{\Omega}^i{}_P(S') = \tilde{\Omega}^i{}_P(S) + \mathrm{d}\alpha.$$

In particular, the class of $\tilde{\Omega}^i{}_P(S)$ in the invariant cohomology with values in $\underline{G}_P/\underline{N}_P$ does not depend on the choice of S in $S(P)$.

Proof. The 1-form $\tilde{\omega}^j{}_P(S') - \tilde{\omega}^j{}_P(S)$ vanishes on vertical vectors and is invariant by the bundle action. Thus it projects to a unique 1-form, α, on $G/G^i{}_P$. Since $\tilde{\omega}^j{}_P(S') - \tilde{\omega}^j{}_P(S)$ is G-invariant and p_{ji} is G-equivariant, uniqueness of α implies that α is G-invariant.

The relation $\tilde{\Omega}^i{}_P(S') - \tilde{\Omega}^i{}_P(S) = \mathrm{d}\alpha$ follows from the fact that the left-hand side is the projection of $\mathrm{d}(\tilde{\omega}^j{}_P(S') - \tilde{\omega}^j{}_P(S))$. □

Until now, we have used the existence of C but not C itself. Since each connected abelian Lie group is isomorphic to the product of a torus by a euclidean space, we can assume that $C(G^i{}_P) = T^p \times \mathbb{R}^q$, $p + q = r(P)$. Thus C gives us an isomorphism from $G^i{}_P/N^j_P$ onto $T^p \times \mathbb{R}^q$, which enables us to identify these groups . In the following the preceding fibre bundle is considered to have $T^p \times \mathbb{R}^q$ as structural group. The Lie algebra of $T^p \times \mathbb{R}^q$ is identified to $\mathbb{R}^{r(P)}$, in such a way that the exponential map becomes

$$Exp(a^1, \dots, a^p, b^1, \dots, b^q) = (\mathrm{e}^{2\pi \mathrm{i} a^1}, \dots, \mathrm{e}^{2\pi \mathrm{i} a^p}, b^1, \dots, b^q).$$

Theorem 7.7. *Let P be completely regular, $i \in I(P)$, $C \in Hom^i{}_P(T^p \times \mathbb{R}^{r(P)-p})$, $N^j_P = KerC$, $S \in S(P)$. Then there exist uniquely defined $\eta^1, \dots, \eta^{r(P)} \in S$ such that $\mathrm{d}C = (\underline{\eta}^1, \dots, \underline{\eta}^{r(P)})$. The connection form is given by $\tilde{\omega}^j{}_P(S) = (\omega^j{}_P(\eta^1), \dots, \omega^j{}_P(\eta^{r(P)}))$ and the curvature form by $\tilde{\Omega}^i{}_P(S) = (\Omega^i{}_P(\eta^1), \dots, \Omega^i{}_P(\eta^{r(P)}))$.*

Proof. ${}^t(\mathrm{d}C)$ gives us an isomorphism from $(\mathbb{R}^{r(P)})^*$ onto $\langle \underline{P} \rangle$. The map defined by sending each element of S to its restriction to $\underline{G}_P$ is an isomorphism from S onto $\langle \underline{P} \rangle$. We denote by f the composite map of the former isomorphism with the inverse of the latter one. To prove existence it suffices to define $\eta^i = f^* e^i$, $i = 1, \dots, r(P)$, where $\{e^i\}$ is the dual basis of the canonical one. Unicity follows from the definition of $S(P)$. □

Let $\sigma \in \langle P \rangle$. Since the restriction of σ to G^0_P is closed, it defines an element, $[\sigma]$, of $H^1(G^0_P, \mathbb{R})$. The image of $H^1(G^0_P, \mathbb{Z})$ under the canonical map into $H^1(G^0_P, \mathbb{R})$ will be denoted also by $H^1(G^0_P, \mathbb{Z})$. We have $[\sigma] \in H^1(G^0_P, \mathbb{Z})$ if and only if $\int_\gamma \sigma$ is an integer for all finite singular 1-cycles with integer coefficients.

Since G_P^0 is a connected Lie group, $\pi_1(G_P^0)$ is abelian so that $\pi_1(G_P^0)$ is canonically isomorphic to $H_1(G_P^0, \mathbb{R})$. Both groups are finitely generated.

In order to prove that $[\sigma] \in H^1(G_P^0, \mathbb{Z})$ it suffices to prove that $\int_{\gamma_i} \sigma \in \mathbb{Z}$, where the γ_i are piecewise differentiable representatives of a set of generators of $\pi_1(G_P^0)$. Let $[\sigma] \in H^1(G_P^0, \mathbb{Z})$. By means of the canonical isomorphism from $Hom(\pi_1(G_P^0), \mathbb{Z})$ into $H^1(G_P^0, \mathbb{Z})$, $[\sigma]$ can be considered as the element of $Hom(\pi_1(G_P^0), \mathbb{Z})$ defined by sending the homotopy class of γ to $\int_\gamma \sigma$.

Let $\eta^1, \dots, \eta^{r(P)}$ be as in Theorem 7.7. Thus $[\eta^k] \in H^1(G_P^0, \mathbb{Z})$, if $k = 1, \dots, p$, and $[\eta^k] = 0$, for $k = p+1, \dots, r(P)$. In fact, as we have seen in the proof of Theorem 7.7, the restrictions of $\eta^1, \dots, \eta^{r(P)}$ to G_P are the pullback by C of the basis dual of the canonical basis of the Lie algebra of $T^p \times \mathbb{R}^{r(P)-p}$. The proof follows from the fact that each of the first p elements of the dual basis has integral 1 on one of the fundamental cycles of $T^p \times \mathbb{R}^{r(P)-p}$ and 0 on the others, whilst the last $r(P) - p$ elements have integral 0 on all the cycles.

Conversely, given a completely regular P, let us assume that there exist $p \le r(P)$ and $\eta^1, \dots, \eta^{r(P)} \in \langle P \rangle$ such that $[\eta^k] \in H^1(G_P^0, \mathbb{R}) - \{0\}$, $k = 1, 2, \dots, p$, $[\eta^k] = 0$, $k = p+1, \dots, r(P)$.

Thus we define $H_P \in Hom_P^0(T^p \times \mathbb{R}^{r(P)-p})$ by sending each $g \in G_P^0$ to

$$H_P(g) = \left(e^{2\pi i \int_\gamma \eta^1}, \dots, e^{2\pi i \int_\gamma \eta^p}, \int_\gamma \eta^{p+1}, \dots, \int_\gamma \eta^{r(P)} \right),$$

where γ is a piecewise differentiable curve in G_P^0 such that $\gamma(0) = e$, $\gamma(1) = g$. Of course, one can give homomorphisms onto other products $T^s \times \mathbb{R}^{r(P)-s}$, $s \ge p$, changing some of the last $R(P) - p$ entries by suitable exponentials.

We see in this way the relation between the search for the nonempty $Hom^i{}_P(T^p \times \mathbb{R}^{r(P)-p})$, and the one for basis of $\langle P \rangle$ composed of elements with integral cohomology class. This relation will be explained with much more detail in the particular case in which P is composed of a single element. This will be accomplished in Section 8.

The preceding considerations applies to the HNDPS in an obvious way: to each completely regular homogeneous nondegenerate 1-system one associates a multihamiltonian space in such a way that one obtains a principal fibre bundle with abelian structural group. Some of the subspaces generated by elements of the 1-system define connections in the principal fibre bundle and the corresponding curvatures are given by linear combinations of elements of the 2-system.

The multihamiltonian spaces associated to equivalent 1-systems are equivalent. Thus we obtain a map from equivalence classes of completely regular homogeneous nondegenerate 1-systems to equivalence classes of completely regular multihamiltonian spaces. This map is not surjective since the multihamiltonian spaces which appear as base spaces of the preceding fibrations are not arbitrary. In fact, we have the following proposition.

Proposition 7.8. *Let η^k be as in Theorem 7.7, $k = 1, 2, \dots, r(P)$. The cohomology class of $\Omega^i{}_P(\eta^k)$ is integral. For $k = p+1, \dots, r(P)$ this cohomology class is 0.*

8. Homogeneous contact manifolds

In this section we consider HNDPS consisting of a single 1-form. As we have seen, its equivalence classes compose a set which is in a one to one correspondence with the subset of $K'(G)$ composed by the $|P, G^i_P, C|$ such that P consists of a single element. This subset will be denoted by $C'(G)$.

Let P be a subset of $\underline{G}^*$ consisting of a single element, σ. Under these circumstances we use the same notation as in the preceding sections but replacing P by σ. In the case where this leads to the use of two σ such as in $\omega^j_\sigma(\sigma)$ or $\Omega^i_\sigma(\sigma)$ one of the σ is suppressed, thus using ω^j_σ or Ω^i_σ, respectively.

Let us denote by $C'_r(G)$ (resp. $C'_{\mathrm{as}}(G)$) the subset of $C'(G)$ composed of $|\sigma, G^i_\sigma, C|$ such that σ is regular (resp. algebraically singular). Obviously, the condition is independent of the representative. $C'(G)$ is the union of $C'_r(G)$ and $C'_{\mathrm{as}}(G)$.

We denote by *Cont*(G) (resp. *Esy*(G)) the set composed of the equivalence classes (in the sense of the HNDPS) of homogeneous contact manifolds (resp. homogeneous exact symplectic spaces). Because of Lemma 3.3, the set of equivalence classes of HNDPS composed of a single 1-form is *Cont*$(G) \cup$ *Esy*(G).

Proposition 8.1. *The map defined by sending* $|\sigma, G^i_\sigma, C|$ *to* $|G/N^j_\sigma, \omega^j_\sigma, G|$, *where* $N^j_\sigma =$ *Ker* C, *maps bijectively* $C'_r(G)$ *onto Cont*(G) *and* $C'_{\mathrm{as}}(G)$ *onto Esy*(G).

Proof. It suffices to prove that no element of the image of $C'_r(G)$ (resp. $C'_{\mathrm{as}}(G)$) is in *Esy*(G) (resp. *Cont*(G)).

If σ is regular, $\mathit{dim}\, \underline{G}_\sigma = \mathit{dim}\, \underline{N}_\sigma + 1$ and $\mathit{dim}\, \underline{G} - \mathit{dim}\, \underline{G}_\sigma$ is even. Thus $\mathit{dim}(G/N^j_\sigma)$ is odd, so that $\mathrm{d}\omega^j_\sigma$ cannot be symplectic.

If σ is algebraically singular, $\mathit{dim}(G/N^j_\sigma)$ is even so that ω^j_σ cannot be a contact form.□

As a consequence of Lemma 7.1 we have the following lemma.

Lemma 8.2. *An element,* σ, *of* $\underline{G}^*$ *is regular if and only if there exists a homomorphism from* G^0_σ *onto* $\mathbf{S}^1$ *whose differential is the restriction of* σ *to* $\underline{G}_\sigma$ *up to a constant factor.*

Let σ be regular. For all $j \in J(\sigma)$, $Z(\omega^j_\sigma)$ (see Example 3.2) is invariant by the action. Since this action is transitive, the group of periods of all its integral curves is the same, and will be denoted in what follows by P^j_σ. This group coincides with $\{t \in \mathbb{R}\colon \operatorname{Exp} tY \in N^j_\sigma\}$, where $Y \in \underline{G}_\sigma$ is such that $\sigma(Y) = 1$. Its nonnegative generator (i.e. the period) will be denoted in what follows by $T(\omega^j_\sigma)$.

Let $|\sigma, G^i_\sigma, C| \in C'_r(G)$. Thus $C(G^i_\sigma)$ is isomorphic to $\mathbb{R}$ or to $\mathbf{S}^1$ and we can assume without loss of generality that one of the following conditions holds:

(i) C is a homomorphism from G^i_σ onto $\mathbb{R}$ such that $\mathrm{d}C$ is the restriction of σ to $\underline{G}^*$.
(ii) C is a homomorphism from G^i_σ onto $\mathbb{R}^1$ such that $\mathrm{d}C$ is, up to a positive factor, the restriction of σ to $\underline{G}^*$.

In fact, there is a representative of the equivalence class, whose homomorphism is in one of the preceding cases.

The subset of $Hom^i_\sigma(\mathbb{R})$ (resp. $\underline{Hom}^i_\sigma(\mathbf{S}^1)$) consisting of the homomorphisms from G^i_σ onto $\mathbb{R}$ (resp. $\mathbf{S}^1$) such that dC is the restriction of σ to $\underline{G}^*$ is denoted in the following by $\underline{Hom}^i_\sigma(\mathbb{R})$ (resp. $\underline{Hom}^i_\sigma(\mathbb{R}^1)$).

Thus in order to obtain all homogeneous contact manifolds, up to equivalence and up to a multiplicative positive constant in the contact form, for a given Lie group, G, one can proceed as follows:

(i) Take a representative of each coadjoint orbit.
(ii) For each representative, σ, determine the corresponding isotropy subgroup G_σ and the G^i_σ, $i \in I(\sigma)$.
(iii) For each i, determine the sets $\underline{Hom}^i_\sigma(\mathbf{S}^1)$ and $\underline{Hom}^i_\sigma(\mathbb{R})$.

The σ that gives rise to homogeneous contact manifold, i.e. the σ such that $\underline{Hom}^i_\sigma(\mathbb{R})$ or $\underline{Hom}^i_\sigma(\mathbb{R}^1)$ are non empty for some $i \in I(\sigma)$, will be called $\mathbb{R}$-*quantizable* or *quantizable*, respectively. Of course, if σ is $\mathbb{R}$-quantizable, it is quantizable. The search for quantizable forms, can be made much easier by developing the methods initiated at the end of Section 7, as follows.

Let $\sigma \in \underline{G}^*$ be such that $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$. Thus, we denote by N_σ the set consisting of the $g \in G^0_\sigma$ such that there exists a curve, γ, in G^0_σ such that $\gamma(0) = e$, $\gamma(1) = g$ and $\int_\gamma \sigma \in \mathbb{Z}$. If $g \in N_\sigma$ and γ' is a curve such that $\gamma'(0) = e$, $\gamma'(1) = g$, we also have $\int_{\gamma'} \sigma \in \mathbb{Z}$. Since $\sigma|_{N^0_\sigma} = 0$ and N^0_σ is connected, $N^0_\sigma \subset N_\sigma$.

Let C_σ be the map from G^0_σ to $\mathbf{S}^1$ defined by sending $g \in G^0_\sigma$ to

$$C_\sigma(g) = \mathrm{e}^{2\pi \mathrm{i} \int_\gamma \sigma},$$

where γ is a piecewise differentiable curve in G^0_σ such that $\gamma(0) = e$, $\gamma(1) = g$. See the definition of H_P in Section 7.

Lemma 8.3. *C_σ is a homomorphism whose kernel is N_σ and* d$C_\sigma = \underline{\sigma}$.

Proof. Let $g, g' \in G^0_\sigma$, and γ, γ' curves such that $\gamma(0) = \gamma'(0) = e$, $\gamma(1) = g$, $\gamma'(1) = g'$. Let $\gamma * g\gamma'$ the "product" (in the homotopy theory sense) of the curve γ by the curve, $g\gamma'$, given by $(g\gamma')(t) = g \cdot (\gamma'(t))$. Hence $(\gamma * g\gamma')(0) = \gamma(0) = e$, $(\gamma * g\gamma')(1) = (g\gamma')(1) = gg'$ and

$$\int_{\gamma * g\gamma'} \sigma = \int_\gamma \sigma + \int_{g\gamma'} \sigma = \int_\gamma \sigma + \int_{\gamma'} \sigma,$$

since $L^*_g \sigma = \sigma$. It follows that $C_\sigma(gg') = C_\sigma(g) C_\sigma(g')$.

The relation $Ker\, C_\sigma = N_\sigma$ is obvious.

For all $X \in \underline{G}_\sigma$, we can consider the curve $\gamma(t) = Exp\, tX$, $t \in [0, 1]$, thus obtaining $C_\sigma(Exp\, X) = \mathrm{e}^{2\pi \mathrm{i} \sigma(X)}$. As a consequence, C_σ is differentiable and d$C_\sigma = \underline{\sigma}$.

In particular, N_σ is closed and N_σ^0 is its connected component of the identity. Thus N_σ^0 is also closed. □

As a consequence, if $[\sigma] \in H^1(G_\sigma^0, \mathbb{Z})$, σ is not topologically singular. If moreover $[\sigma] \neq 0$, σ is regular.

Now let us assume that $[\sigma] = 0$. Thus we denote by C'_σ the map from G_σ^0 into $\mathbb{R}$, defined by sending $g \in G_\sigma^0$ to

$$C'(g) = \int_\gamma \sigma,$$

where γ is a curve in G_σ^0 such that $\gamma(0) = e$, $\gamma(1) = g$. See the definition of H_P in Section 7.

Lemma 8.4. *C'_σ is a homomorphism whose kernel is N_σ^0 and* $\mathrm{d}C'_\sigma = \underline{\sigma}$.

Proof. We see, as in the proof of Lemma 8.3., that C'_σ is a homomorphism and $C'_\sigma(Exp\, tX) = t\sigma(X)$ for all $t \in \mathbb{R}$, $X \in \underline{G_\sigma}$. It follows that C'_σ is a Lie group homomorphism and that $\mathrm{d}C'_\sigma = \underline{\sigma}$. In particular $N_\sigma^0 \subset Ker\, C'_\sigma$.

If σ is algebraically singular $Ker\, C'_\sigma \subset G_\sigma^0 = N_\sigma^0$ and the proof is complete.

Now let us assume that σ is not algebraically singular. Thus, there exists $Y \in \underline{G_\sigma}$ such that $\sigma(Y) = 1$. Because of Lemma 5.8, for all $h \in G_\sigma^0$, there exists $t \in \mathbb{R}$, $n \in N_\sigma^0$, such that $h = (Exp\, tY)n$. Thus $t = C'_\sigma(h)$. In particular, if $h \in Ker\, C'_\sigma$ we have $t = 0$ so that $h \in N_\sigma^0$. □

We quote here for future reference the following lemma.

Lemma 8.5. *Let $[\sigma] \in H^1(G_\sigma^0, \mathbb{Z})$, $g \in G_\sigma$, $g_0 \in G_\sigma^0$. Thus $g_0^{-1}g^{-1}g_0g \in N_\sigma^0$.*

Proof. Let γ be a curve in G_σ^0 such that $\gamma(0) = e$ and $\gamma(1) = g_0$. Since $L^*_{g^{-1}}R^*_g\sigma = \sigma$, we have

$$\int_\gamma \sigma = \int_\gamma L^*_{g^{-1}}R^*_g\sigma = \int_{g^{-1}\gamma g} \sigma$$

so that $C_\sigma(g_0) = C_\sigma(g^{-1}g_0g)$. Hence we have $g_0^{-1}g^{-1}g_0g \in N_\sigma$. Thus $g_0^{-1}g^{-1}g_0g$ can be joined to e by a curve contained in N_σ. Hence $g_0^{-1}g^{-1}g_0g \in N_\sigma^0$. □

Corollary 8.6. *If $[\sigma] \in H^1(G_\sigma^0, \mathbb{R})$, then N_σ^0 and N_σ are invariant subgroups of G_σ.*

Let $[\sigma] \in H^1(G_\sigma^0, \mathbb{Z})$, $i \in I(\sigma)$. Let p_σ^i (resp. $p_\sigma'^i$) be the canonical map from G_σ^i/N_σ (resp. G_σ^i/N_σ^0) onto G_σ^i/G_σ^0. We denote by S_σ^i (resp. R_σ^i) the set composed of the subgroups, N, of G_σ^i/N_σ (resp. G_σ^i/N_σ^0) such that $p_\sigma^i|_N$ (resp. $p_\sigma'^i|_N$) is bijective, i.e. an isomorphism.

Notice that S_σ^i (resp. R_σ^i) is bijective to the set composed of those sections of p_σ^i (resp. $p_\sigma'^i$) which are homomorphisms.

Theorem 8.7. *Let σ be regular. The following conditions are equivalent:* (1) $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$; (2) $T(\omega^0_\sigma) \in \mathbb{Z}$; (3) $\underline{Hom}^0_\sigma(\mathbf{S}^1) \neq \emptyset$. *If these conditions hold,* $\underline{Hom}^i_\sigma(\mathbf{S}^1)$ *is bijective to* S^i_σ *and, if non-empty, to* $Hom(G^i_\sigma / G^0_\sigma, \mathbf{S}^1)$, *for all* $i \in I(\sigma)$.

Proof. Since σ is regular, there exists $Y \in \underline{G}_\sigma$ such that $\sigma(Y) = 1$. If $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$, we have $C_\sigma(Exp\, tY) = e^{2\pi i t}$ and $Exp\, T(\omega^0_\sigma)Y \in N^0_\sigma \subset N_\sigma$. Thus $T(\omega^0_\sigma) \in \mathbb{Z}$.

Now, let us assume that $T(\omega^0_\sigma) \in \mathbb{Z}$. Given $g_0 \in G^0_\sigma$, let $t \in \mathbb{R}$, $n \in N^0_\sigma$ be such that $g_0 = (Exp\, tY)n$. The real number t is well defined modulo $T(\omega^0_\sigma)$, by g_0. The map defined by sending g_0 to $e^{2\pi i t}$ is well defined and an element of $\underline{Hom}^0_\sigma(\mathbf{S}^1)$.

Now let $i \in I(\sigma)$, $C \in \underline{Hom}^0_\sigma(\mathbf{S}^1)$. Let μ be the left invariant 1-form on $\mathbf{S}^1$ such that $\mu(1) = 1$. $C^*\mu$ is a left invariant 1-form on G^0_σ such that $C^*\mu(Z) = \mu(\mathrm{d}C(Z)) = \mu(\sigma(Z)) = \sigma(Z)$ for all $Z \in \underline{G}_\sigma$. Thus $C^*\mu = \sigma$ and, since $[\mu] \in H^1(\mathbf{S}^1, \mathbb{Z})$, it follows that $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$. This finishes the proof of the equivalence of the three conditions.

Now let us assume that the conditions hold. We shall define a one to one map from S^i_σ onto $\underline{Hom}^i_\sigma(\mathbf{S}^1)$.

Let $N \in S^i_\sigma$. We define a map, f_N, from G^i_σ onto G^0_σ / N_σ by means of $f_N(g) = g((p^i_\sigma|_N)^{-1}(g^{-1}G^0_\sigma))$. The map f_N is an onto homomorphism.

On the other hand C_σ gives us an isomorphism, $\underline{C}_\sigma$, from G^0_σ / N_σ onto $\mathbf{S}^1$ (since σ is not algebraically singular and $\mathrm{d}C_\sigma = \underline{\sigma}$, C_σ is surjective). Let us denote by C_N the composite map $\underline{C}_\sigma \circ f_N$. Since the restriction of C_N to G^0_σ is C_σ, we have $\mathrm{d}C_N = \underline{\sigma}$. Thus $C_N \in \underline{Hom}^i_\sigma(\mathbf{S}^1)$.

We shall prove that the map, φ^i_σ, from S^i_σ into $\underline{Hom}^i_\sigma(\mathbf{S}^1)$ defined by $\varphi^i_\sigma(N) = C_N$ is bijective. We shall explicitly give its inverse.

Let $C \in \underline{Hom}^i_\sigma(\mathbf{S}^1)$. We have $C|_{G^0_\sigma} = C_\sigma$ so that $N_\sigma = G^0_\sigma \cap Ker\, C$.

We have $Ker\, C / N_\sigma \in S^i_\sigma$. In fact, for all $g \in G^i_\sigma$, there exists $g_0 \in G^0_\sigma$, $n \in Ker\, C$, such that $g = ng_0$. Thus $gG^0_\sigma = nG^0_\sigma = p^i_\sigma(nN_\sigma)$. This proves that $(p^i_\sigma|_H)$ is an onto map, where we have denoted $Ker\, C / N_\sigma$ by H. On the other hand, if $n, n' \in Ker\, C$ are such that $nG^0_\sigma = n'G^0_\sigma$, then we have $n^{-1}n' \in G^0_\sigma \cap Ker\, C = N_\sigma$. This proves that $p^i_\sigma|_H$ is injective.

The map defined by sending $C \in \underline{Hom}^i_\sigma(\mathbf{S}^1)$ to $Ker\, C / N_\sigma \in S^i_\sigma$, is the inverse of φ^i_σ.

To end the proof, notice that $Hom(G^i_\sigma / G^0_\sigma, \mathbf{S}^1)$ acts freely and transitively on $\underline{Hom}^i_\sigma(\mathbf{S}^1)$, if this set is nonempty. The action is given by $C \cdot F(g) = C(g)F(gG^0_\sigma)$ for all $F \in Hom(G^i_\sigma / G^0_\sigma, \mathbf{S}^1)$, $C \in \underline{Hom}^i_\sigma(\mathbf{S}^1)$ and $g \in G^i_\sigma$. □

Corollary 8.8. *Let σ be regular. Thus there exists* $A \in \mathbb{R} - \{0\}$ *such that* $A[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$.

Proof. If σ is regular, there exists $A \in \mathbb{R} - \{0\}$ such that $\underline{Hom}^0_{A\sigma}(\mathbf{S}^1) \neq \emptyset$ (cf. Lemma 8.2). Thus $[A\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$. □

As a consequence of this result and the remark following the proof of Lemma 8.3, we have the following corollary.

Corollary 8.9. *Let us assume that σ is not algebraically singular. σ is regular if and only if* $[A\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$ *for some* $A \in \mathbb{R} - \{0\}$.

Theorem 8.10. *Let σ be regular. The following conditions are equivalent:* (1) $[\sigma] = 0$; (2) $T(\omega^0_\sigma) = 0$; (3) $\underline{Hom}^0_\sigma(\mathbb{R}) \neq \emptyset$. *If these conditions hold,* $\underline{Hom}^i_\sigma(\mathbb{R})$ *is bijective to* R^i_σ *and, if nonempty, to* $Hom(G^i_\sigma/G^0_\sigma, \mathbb{R})$ *for all* $i \in I(\sigma)$.

The proof of this theorem is similar to that of Theorem 8.7, and is left to the reader.

As a consequence, if σ is regular and $[\sigma]$ is integral but not zero, $\underline{Hom}^i_\sigma(\mathbb{R})$ is empty for all i.

Notice that, if G is simply connected G^i_σ/G^0_σ is isomorphic to the fundamental group of the Hamiltonian space G/G^i_σ, so that $Hom(G^i_\sigma/G^0_\sigma, H)$ is isomorphic to $Hom(\pi_1(G/G^i_\sigma), H)$.

Let σ be regular and $[\sigma] = 0$. Let s be a section of the canonical map from G^i_σ onto G^i_σ/G^0_σ. For all $K, K' \in G^i_\sigma/G^0_\sigma$, there exists a uniquely defined $\bar{s}(K, K') \in \mathbb{R}$ such that $s(K)\cdot s(K') = s(K\cdot K')(Exp\,\bar{s}(K, K')Y)n$, where $n \in N^0_\sigma$ and $Y \in \underline{G}$ such that $\sigma(Y) = 1$. We thus obtain a 2-cochain, $\bar{s} \in C^2(G^i_\sigma/G^0_\sigma, \mathbb{R})$, which by direct calculation is proved to be a 2-cocycle. The corresponding cohomology class, $[\bar{s}] \in H^2(G^i_\sigma/G^0_\sigma, \mathbb{R})$ is independent of the section we have chosen and will be denoted by t^i_σ. If we identify G^0_σ/N^0_σ with $\mathbf{S}^1$ by means of C'_σ, t^i_σ is the cohomology class corresponding to the extension of G^i_σ/G^0_σ by G^i_σ/N^0_σ.

We shall also denote by $H^2(G^i_\sigma/G^0_\sigma, \mathbb{Z})$ the image of the natural homomorphism of $H^2(G^i_\sigma/G^0_\sigma, \mathbb{Z})$ into $H^2(G^i_\sigma/G^0_\sigma, \mathbb{R})$.

Proposition 8.11. *Let σ be regular and $[\sigma] = 0$. We have* $\underline{Hom}^i_\sigma(\mathbf{S}^1) \neq \emptyset$ *(resp.* $\underline{Hom}^i_\sigma(\mathbb{R}) \neq \emptyset$*) if and only if* $t^i_\sigma \in H^2(G^i_\sigma/G^0_\sigma, \mathbb{Z})$ *(resp.* $t^i_\sigma = 0$*).*

Proof. Let $N \in S^i_\sigma$ (resp. $N \in R^i_\sigma$) and let us choose a section s in such a way that $s(K)N_\sigma \in N$ (resp. $s(K)N^0_\sigma \in N$) for all $K \in G^i_\sigma/G^0_\sigma$. Thus, for all $K, K' \in G^i_\sigma/G^0_\sigma$, we have $(Exp\,\bar{s}(K, K')Y)n = s(K\cdot K')^{-1}s(K)s(K') \in N_\sigma$ (resp. N^0_σ), since N is a subgroup of G^i_σ/N_σ (resp. G^i_σ/N^0_σ).

On the other hand $C'_\sigma(Exp\,tY)n) = t$. Thus $\bar{s}(K, K') \in \mathbb{Z}$ (resp. $\bar{s}(K, K') = 0$).

Conversely, let $t^i_\sigma \in H^2(G^0_\sigma, \mathbb{Z})$(resp. $t^i_\sigma = 0$). Let s be a section such that $\bar{s}(K, K') \in \mathbb{Z}$ (resp. $s = \delta H$, where $H \in C^1(G^i_\sigma/G^0_\sigma, \mathbb{R})$). Thus $\{s(K)N_\sigma : K \in G^i_\sigma/G^0_\sigma\} \in S^i_\sigma$ (resp. $\{s(K)(Exp{-H}(K)Y)N^0_\sigma : K \in G^i_\sigma/G^0_\sigma\} \in R^i_\sigma$). □

As a consequence, if $H^2(G^i_\sigma/G^0_\sigma, \mathbb{R}) = 0$, then $\underline{Hom}^i_\sigma(\mathbb{R}) \neq \emptyset$. If the subset $H^2(G^i_\sigma/G^0_\sigma, \mathbb{Z})$ of $H^2(G^i_\sigma/G^0_\sigma, \mathbb{R})$ vanishes and $\underline{Hom}^i_\sigma(\mathbf{S}^1) \neq \emptyset$, then $\underline{Hom}^i_\sigma(\mathbb{R}) \neq \emptyset$.

Corollary 8.12. *If G^i_σ has a finite number of components, then* $\underline{Hom}^i_\sigma(\mathbb{R})$ *consists of exactly one element.*

Proof. Let m be the order of G^i_σ/G^0_σ. Multiplication by m in $\mathbb{R}$ is an isomorphism, thus giving an isomorphism in cohomology. But multiplication by m is the zero homomorphism in cohomology (cf. [5, 16.5, p.227]). Thus $H^n(G^i_\sigma/G^0_\sigma, \mathbb{R}) = 0$ for all $n \geq 1$. Thus $\underline{Hom}^i_\sigma(\mathbb{R})$ is nonempty so that it is bijective to $Hom(G^i_\sigma/G^0_\sigma, \mathbb{R})$. But $Hom(G^i_\sigma/G^0_\sigma, \mathbb{R})$

consists only of the identity element. □

Also in the general case in which σ is regular and $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$ we have cohomological methods to know the cardinality of the sets $\underline{Hom}^i_\sigma(\mathbf{S}^1)$.

In this case, we can consider the action of G^i_σ/G^0_σ on $\mathbf{S}^1$ by automorphisms (i.e. $\mathbf{S}^1$ becomes a (G^i_σ/G^0_σ)-module) given by $gG^0_\sigma * C_\sigma(g_0) = C_\sigma(gg_0g^{-1})$ and the cochain defined by sending a pair $(k, k') \in (G^i_\sigma/G^0_\sigma)^2$ to $C_\sigma(s(k)s(k')s(k,k')^{-1})$, where s is a section of the canonical map of G^i_σ onto G^i_σ/G^0_σ. This cochain is in fact a cocycle and its cohomology class, $\tau^i_\sigma \in H^2(G^i_\sigma/G^0_\sigma, \mathbf{S}^1)$, does not depend on the section s. If we identify G^0_σ/N_σ with $\mathbf{S}^1$ by means of C_σ, τ^i_σ is the cohomology class corresponding to the extension of G^i_σ/G^0_σ by G^i_σ/N_σ. We have the following proposition.

Proposition 8.13. *$\underline{Hom}^i_\sigma(\mathbf{S}^1)$ is nonempty if and only if $\tau^i_\sigma = 0$. When this is the case, $\underline{Hom}^i_\sigma(\mathbf{S}^1)$ is bijective with the set composed by the 1-cocycles, $Z^1(G^i_\sigma/G^0_\sigma, \mathbf{S}^1)$.*

Let $[\sigma] \in H_1(G^0_\sigma, \mathbb{Z})$. The image of $[\sigma]$, $[\sigma](H_1(G^0_\sigma, \mathbb{Z}))$, is a subgroup of $\mathbb{Z}$ whose nonnegative generator is the highest common factor of the subgroup if $[\sigma] \neq 0$ and 0 if $[\sigma] = 0$. If $[\sigma] \neq 0$, it is also the highest common factor of the $\int_{\gamma_i} \sigma$, where γ_i, $i = 1, \ldots, N$, are piecewise differentiable curves whose homotopy classes generate $\pi_1(G^0_\sigma, e)$.

Proposition 8.14. *Let σ be regular and $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z})$. Thus the nonnegative generator of $[\sigma](H_1(G^0_\sigma, \mathbb{Z}))$ is $T(\omega^0_\sigma)$, i.e. $[\sigma](H_1(G^0_\sigma, \mathbb{Z})) = P^0_\sigma$.*

Proof. If $[\sigma] = 0$, we know from the preceding theorem that $T(\omega^0_\sigma) = 0$.

If $[\sigma] \neq 0$, we have $\underline{Hom}^0_\sigma(\mathbb{R}) = \emptyset$. Thus G^0_σ/N^0_σ must be isomorphic to $\mathbf{S}^1$. Let f be an isomorphism from G^0_σ/N^0_σ onto $\mathbf{S}^1$. Thus $\mathrm{d}(f \circ C^0_\sigma) = A\underline{\sigma}$ for some $A \in \mathbb{R} - \{0\}$. We can choose f so that $A \in \mathbb{R}^+$. We thus have $A = 1/T(\omega^0_\sigma)$, so that $f \circ C^0_\sigma \in \underline{Hom}^0_{\sigma/T(\omega^0_\sigma)}(\mathbf{S}^1)$. In particular, $[\sigma/T(\omega^0_\sigma)] \in H^1(G^0_\sigma, \mathbb{Z})$, so that $T(\omega^0_\sigma)$ is a common factor of the $\int_\gamma \sigma$ for all piecewise differentiable curves in G^0_σ starting and ending at e.

Let us denote by γ_1 the curve $\gamma_1(t) = Exp\, tT(\omega^0_\sigma)Y$, $t \in [0, 1]$, where $Y \in \underline{G_\sigma}$ is such that $\sigma(Y) = 1$. Let γ_2 be a curve in N^0_σ starting at $Exp\, T(\omega^0_\sigma)Y$ and ending at e. Thus the integral of σ on $\gamma_1 * \gamma$ is $T(\omega^0_\sigma)$, so that $T(\omega^0_\sigma)$ is the highest common factor of the $\int_\gamma \sigma$. □

Corollary 8.15. *Let $[\sigma] \in H^1(G^0_\sigma, \mathbb{Z}) - \{0\}$. Thus $N_{\sigma/T(\omega^0_\sigma)} = N^0_\sigma$.*

Proof. Let $Y \in \underline{G_\sigma}$ be such that $\sigma(Y) = 1$. Thus $N_\sigma = \{(Exp\, tY)n : t \in \mathbb{Z}, n \in N^0_\sigma\}$. Since $[\sigma/T(\omega^0)] \in H^1(G^0_\sigma, \mathbb{Z})$ we can consider $N_{\sigma/T(\omega^0_\omega)}$ and we have $N_{\sigma/T(\omega^0_\sigma)} = \{(Exp\, tT(\omega^0_\sigma)Y)n : t \in \mathbb{Z}, n \in N^0_\sigma\} = N^0_\sigma$. □

As a consequence of these results, one can proceed as follows in order to know which 1-forms are quantizable. After steps (i) and (ii) have been completed, one can look for a set of generators of $\pi_1(G^0_\sigma)$ and evaluate the integral of σ along all of them. Then:

(i) If these integrals are all zero, σ and all of its proportionals are $\mathbb{R}$-quantizable and, as a consequence, quantizable. The concrete $i \in I(\sigma)$ for which the homomorphisms exist, can be determined by means of R^i_σ or t^i_σ.

(ii) If there exists a $\lambda \in \mathbb{R}$ whose product by the integrals are integral numbers, not all zero, let us denote by T the greatest common factor of these integers. Thus all the $k\lambda\sigma/T$, $k \in \mathbb{Z}$, are quantizable but not $\mathbb{R}$-quantizable. The concrete $i \in I(\sigma)$ for which the homomorphisms exist, can be determined by means of S^i_σ or τ^i_σ.

(iii) If no such a λ exists, then no 1-form proportional to σ is quantizable.

Once the quantizable forms are known, one can determine C_σ and C'_σ (when it exists) and then, complete step (iii).

The HCM, like all HNDPS (cf. Section 7), gives rise to principal fibre bundles with connection. We now describe briefly the situation in this particular case. More details are given in [4].

Let σ be regular, $\underline{G} \neq \underline{G_\sigma}$, $i \in I(\sigma)$ and $C \in \underline{Hom^i_\sigma}(\mathbf{S}^1)$.

We can define an action of $\mathbf{S}^1$ on $G/Ker\,C$ by means of $(g\,Ker\,C)*s = gh\,Ker\,C$, where h is an element of G^i_σ such that $C(h) = s$. Actually $(G/Ker\,C)(G/G^i_\sigma, \mathbf{S}^1)$ is a principal fibre bundle, the bundle action being the preceding one and the bundle projection, the canonical map.

Let $Z(\omega)$ be the vector field defined by

$$i_{Z(\omega)}\omega = 1, \qquad i_{Z(\omega)}\mathrm{d}\omega = 0,$$

where ω is the contact form defined on $G/Ker\,C$ by σ. If we denote by $T(\omega)$ the period of any integral curve of $Z(\omega)$, then we have $\lambda = 1/T(\omega)$. Thus $\omega/T(\omega)$ is a connection form. Since the structural group is abelian, the curvature form is $\mathrm{d}\omega/T(\omega)$. There exists a unique 2-form on G/G^i_σ, whose pullback under the bundle map is the curvature form. This form itself will also be called curvature form. Its reciprocal image under the canonical map of G onto G/G^i_σ is $\mathrm{d}\sigma/T(\omega)$. This form is symplectic and its cohomology class is integral. The manifolds G/G^i_σ provided with these symplectic structures are the Hamiltonian spaces (cf. [7]) of the group G.

If $C \in Hom^i_\sigma(\mathbb{R})$ we have similar results but the structural group is $\mathbb{R}$ and the connection form is, simply, the projection of σ.

In any case, the manifold $G/Ker\,C$ is the quotient of G/N^0_σ by a properly discontinuous free action of $Ker\,C/N^0_\sigma$. The following lemmas give us information regarding this group and the action. For each quantizable form, we consider all of its proportionals which are quantizable.

Lemma 8.16. *Let σ be regular, $T(\omega^0_\sigma) \in \mathbb{Z}-\{0\}$ and $C \in \underline{Hom}^i_{k\sigma/T(\omega^0_\sigma)}(\mathbf{S}^1)$, $k \in \mathbb{Z}-\{0\}$. Thus we have an exact sequence $1 \to \mathbb{Z}/k\mathbb{Z} \to Ker\,C/N^0_\sigma \to G^i_\sigma/G^0_\sigma \to 1$.*

Proof. Let $Y \in \underline{G_\sigma}$ be such that $\sigma(Y) = 1$. Since $C((Exp\,tY)n) = \mathrm{e}^{2\pi i tk/T(\omega^0_\sigma)}$ for all $t \in \mathbb{R}, n \in N^0_\sigma$, we see that $G^0_\sigma \cap Ker\,C = \{(Exp(zT(\omega^0_\sigma)/k)Y)n: z \in \mathbb{Z}, n \in N^0_\sigma\}$.

The map, f, defined by sending $z + k\mathbb{Z}$ to $(Exp(zT(\omega^0_\sigma)/k)Y)N^0_\sigma \in Ker\,C/N^0_\sigma$ is a well defined injective homomorphism whose image is $((Ker\,C) \cap G^0_\sigma)/N^0_\sigma$. Thus the sequence

$1 \to \mathbb{Z}/k\mathbb{Z} \xrightarrow{f} Ker\,C/N_\sigma^0 \xrightarrow{p} Ker\,C/(G_\sigma^0 \cap Ker\,C) \to 1$, where p is the canonical map, is exact. But $\ker C/(G_\sigma^0 \cap Ker\,C) \in S^i_{k\sigma/T(\omega_\sigma^0)}$ (cf. the proof of Theorem 8.7) so that it is isomorphic to G_σ^i/G_σ^0. □

Lemma 8.17. *Let σ be regular, $T(\omega_\sigma^0) = 0$ and $C \in Hom_\sigma^i(H)$, where $H = \mathbb{R}$ or $H = \mathbf{S}^1$. If $H = \mathbb{R}$, $Ker\,C/N_\sigma^0 \simeq G_\sigma^i/G_\sigma^0$, if $H = \mathbf{S}^1$, we have an exact sequence $1 \to \mathbb{Z} \to Ker\,C/N_\sigma^0 \to G_\sigma^i/G_\sigma^0 \to 1$.*

Proof. Let $A \in \mathbb{R} - \{0\}$ be such that $C \in \underline{Hom}^i_{A\sigma}(H)$. If $H = \mathbb{R}$, we have $C/A \in \underline{Hom}^i_\sigma(\mathbb{R})$. Thus $(Ker\,C)/N_\sigma^0 = (Ker(C/A))/N_\sigma^0 \in R_\sigma^i$ and the result follows. If $H = \mathbf{S}^1$, we have $C((Exp\,tY)n) = e^{2\pi i A t}$, for all $t \in \mathbb{R}, n \in N_\sigma^0$. Thus $G_\sigma^0 \cap Ker\,C = \{(Exp(z/A)Y)n : z \in \mathbb{Z}, n \in N_\sigma^0\}$. The remainder of the proof is similar to that of Lemma 8.16. □

Proposition 8.18. *Let σ be regular and $T(\omega_\sigma^0) \in \mathbb{Z} - \{0\}$. Then G/N_σ is the quotient of G/N_σ^0 by the properly discontinuous free action of the group of $T(\omega_\sigma^0)$-roots of 1 given by the bundle $\mathbf{S}^1$-action.*

Proof. The diffeomorphism of G/N_σ^0 associated by the action to $gN_\sigma^0 \in N_\sigma/N_\sigma^0$ is the diffeomorphism associated by the bundle action to gN_σ^0 considered as an element of G_σ^0/N_σ^0 i.e. the diffeomorphism associated by the bundle action to $C_{\sigma/T(\omega_\sigma^0)}(g) \in \mathbf{S}^1$ (cf. Corollary 8.15). Since $N_\sigma = \{(Exp\,tY)n : t \in \mathbb{Z}, n \in N_\sigma^0\}$, we have $C_{\sigma/T(\omega_\sigma^0)}(N_\sigma) = \{e^{2\pi i t/T(\omega_\sigma^0)} : t \in \mathbb{Z}\}$. □

In a similar way we have the following proposition.

Proposition 8.19. *Let σ be regular and $T(\omega_\sigma^0) = 0$. Then G/N_σ is the quotient of G/N_σ^0 by the properly discontinuous free action of $\mathbb{Z}$ given by the bundle $\mathbb{Z}$-action.*

9. An example: Quantizable forms for relativistic particles

It is a generally accepted fact that relativistic free particles correspond, via Geometric Quantization, to quantizable forms of the universal covering group of Poincaré Group. In this section we apply the preceding methods to this particular case.

The universal covering group of Poincaré group is the semidirect product $\mathbf{SL}(2, \mathbb{C}) \oplus \mathbf{H}(2)$, with group law given by $(A, H) * (B, K) = (AB, AKA^* + H)$.

The Lie algebra of $\mathbf{SL} \oplus \mathbf{H}(2)$ is identified to $\mathbf{sl} \times \mathbf{H}(2)$, the Lie bracket being

$$[(a, k), (a', k')] = ([a, a'], ak' + k'a^* - (a'k + ka'^*)).$$

The adjoint representation thus becomes

$$Ad_{(A, H)}(a, k) = (AaA^{-1}, AkA^* - (AaA^{-1})H - H(AaA^{-1})^*).$$

Table 1
Types of coadjoint orbits

Type	$\|P\|$	$\|W\|$	P	W	$Det\, a$
1	0	0	0	0	0
2	0	0	0	0	$\neq 0$
3	0	<0	$\neq 0$	$\neq 0$	
4	0	0	$\neq 0$		
5	>0	≤ 0	$\neq 0$		
6	<0	0	$\neq 0$	0	
7	<0	>0	$\neq 0$	$\neq 0$	
8	<0	<0	$\neq 0$	$\neq 0$	
9	<0	0	$\neq 0$	$\neq 0$	

We define a nondegenerate scalar product in $\mathbf{sl} \times \mathbf{H}(2)$ by means of

$$\langle (a, k), (b, l) \rangle = -2\, Re\, Tr(\tfrac{1}{4} k \sigma_2 \bar{l} \sigma_2 + ab).$$

This scalar product defines in the standard way an isomorphism from the Lie algebra of $\mathbf{SL} \oplus \mathbf{H}(2)$ onto its dual. The image of $(a, k) \in \mathbf{sl} \times \mathbf{H}(2)$ will be denoted by $\{a, k\}$.

With this notation, we obtain by a more or less straightforward computation, the following formula for the coadjoint representation

$$Ad^*_{(A,H)}\{a, k\} = \{AaA^{-1} + \tfrac{1}{4}(AkA^* \varepsilon \overline{H} \varepsilon - H \varepsilon \overline{AkA^*} \varepsilon), AkA^*\}.$$

Now we shall describe a canonical choice of representative of each coadjoint orbit. To do that, it is useful to introduce functions which remain constant along coadjoint orbits.

One of these is $|P|$, defined by $|P|\,(\{a, k\}) = Det(k)$, and is usually interpreted as being the mass square.

Another of these functions is obtained from $W(\{a, k\}) = i(a\,k - k\,a^*)$. One can prove that $W(Ad^*_{(A,H)}\{a,\ k\}) = A\ W(\{a, k\})\ A^*$, so that $|W|(\{a, k\}) = Det(W(\{a, k\}))$, is constant on each coadjoint orbit. The four-vector corresponding to W is the Pauly–Lubanski one.

Tables 1 and 2 give a classification of the coadjoint orbits other than $\{0\}$. They are divided into nine different types, which are numbered in the first column.

Each coadjoint orbit has a unique representative of those considered in the second column of Table 2. The form we have given to these representatives, is designed to ease the task of finding the canonical representative of the orbit of any given element. In fact, when $\{a, k\}$ is given, the values of $|P|$ and $|W|$ and the nullity or not of P and W (and $Det\, a$ in the case where all the others values are zero) determine its type by means of columns 2–6 of Table 1. Then, with these values and the conditions in the third column of Table 2, one finds the canonical representative of the coadjoint orbit of $\{a, k\}$.

The application of the methods of Section 8 to each of these representatives, leads to the following results.

There are three singular orbits (i.e. orbits consisting of singular forms) other than $\{0\}$. Its canonical representatives are

Table 2
Canonical representatives

Type	Representative	Conditions
1	$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right\}$	
2	$\left\{\sqrt{-Det\, a}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right\}$	$Im\left(\sqrt{-Det\, a}\right) > 0$ or $\sqrt{-Det\, a} \in \mathbb{R}^+$
3	$\left\{\sqrt{-\lvert W\rvert}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, -sig(Tr(P))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$	$\sqrt{-\lvert W\rvert} \in \mathbb{R}^+$
4	$\left\{\frac{\mathrm{i}s}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -sig(Tr(P))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$	$W = sP$
5	$\left\{\frac{\mathrm{i}}{2}\sqrt{\frac{-\lvert W\rvert}{\lvert P\rvert}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -sig(Tr(P))\sqrt{\lvert P\rvert}\, I\right\}$	$\sqrt{\frac{-\lvert W\rvert}{\lvert P\rvert}} \in \mathbb{R}^+ \cup \{0\}$, $\sqrt{\lvert P\rvert} \in \mathbb{R}^+$
6	$\left\{0, \sqrt{-\lvert P\rvert}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$	$\sqrt{-\lvert P\rvert} \in \mathbb{R}^+$
7	$\left\{\frac{-\mathrm{i}}{2} sig(Tr(W))\sqrt{\frac{-\lvert W\rvert}{\lvert P\rvert}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-\lvert P\rvert}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$	$\sqrt{\frac{-\lvert W\rvert}{\lvert P\rvert}}$, $\sqrt{-\lvert P\rvert} \in \mathbb{R}^+$
8	$\left\{\sqrt{\frac{\lvert W\rvert}{\lvert P\rvert}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sqrt{-\lvert P\rvert}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$	$\sqrt{\frac{\lvert W\rvert}{\lvert P\rvert}}$, $\sqrt{-\lvert P\rvert} \in \mathbb{R}^+$
9	$\left\{\begin{pmatrix} \eta\mathrm{i} & 1 \\ 1 & -\eta\mathrm{i} \end{pmatrix}, \sqrt{-\lvert P\rvert}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$	$\sqrt{-\lvert P\rvert} \in \mathbb{R}^+$, $\eta \in \{-1, +1\}$

$$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right\} \text{ (type 1)} \quad \text{and} \quad \left\{0,\ \eta\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\} \text{ (type 4)},$$

where $\eta = 1, -1$. $\mathbb{R}$-quantizable orbits are all of types 3, 6, 8, 9 and those of type 5 corresponding to the case $|W| = 0$

Quantizable but no $\mathbb{R}$-quantizable are the orbits whose canonical representatives are

$$\left\{\frac{\mathrm{i}T}{8\pi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right\} \quad \text{(type 2)},$$

$$\left\{\frac{\mathrm{i}\chi T}{8\pi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -sign(Tr(P))\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\} \quad \text{(type 4)},$$

$$\left\{\frac{iT}{8\pi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -sign(Tr(P))\sqrt{|P|}\, I\right\} \quad \text{(type 5)},$$

$$\left\{\frac{i\chi T}{8\pi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \sqrt{-|P|}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\} \quad \text{(type 7)},$$

where, in all cases, $T \in \mathbb{Z}^+$, $\chi = 1, -1$.

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